Disease Eradication if $\mathcal{R}_0 < 1$? It's Not That Simple (Math 747 Weekly Update)

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Selected paper

- "Occurrence of backward bifurcation and prediction of disease transmission with imperfect lockdown: A case study on COVID-19" by Sk Shahid Nadim & Joydev Chattopadhyay. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC7430254/
- Topic: Develop and analyze compartmental model for COVID-19 to account for lockdown efficacy and success rate.

Discussion overview

1. Introduce model
2. Key insights and findings
3. Conclusions and takeaways

Introducing model

The model proposed by Nadim and Chattopadhyay is a compartmental model. Unlike the standard SEIR model with vital dynamics, here, we divide the population into 6 mutually exclusive groups:

S - Susceptible,
L - Lockdown,
E - Exposed,
I - Infected (un-notified),
J - Hospitalized/Isolated,
R - Recovered,

Total population $N(t) = S(t) + L(t) + E(t) + I(t) + J(t) + R(t)$.

System Parameters

The model includes several additional parameters. We pay close attention to r, l, and $1/\psi$; key differences from the SEIR model with vital dynamics.

Π – Recruitment rate of human population (source of new susceptibles)

$1/\mu$ – Average life expectancy at birth

β – Transmission rate of infected individuals

r – Lockdown efficacy (perfect $\implies r = 0$, imperfect $\implies 0 < r < 1$)

$1/\gamma$ – COVID-19 incubation period (estimate range 1 - 14 days)

l – Lockdown success rate (range 0 - 1)

$1/\psi$ – Lockdown period

η – Rate at which symptomatic infected become hospitalized/notified

δ – Death rate of hospitalized/notified population

τ_1 – Recovery rate for symptomatic infected

τ_2 – Recovery rate for hospitalized/notified individuals

ODEs

The model is a deterministic one, and it's implemented via the following system of ordinary differential equations.

$$\frac{dS}{dt} = \Pi + \psi L - \frac{\beta S I}{N - J} - (\mu + l)S,$$

$$\frac{dL}{dt} = lS - \frac{r\beta L I}{N - J} - (\mu + \psi)L,$$

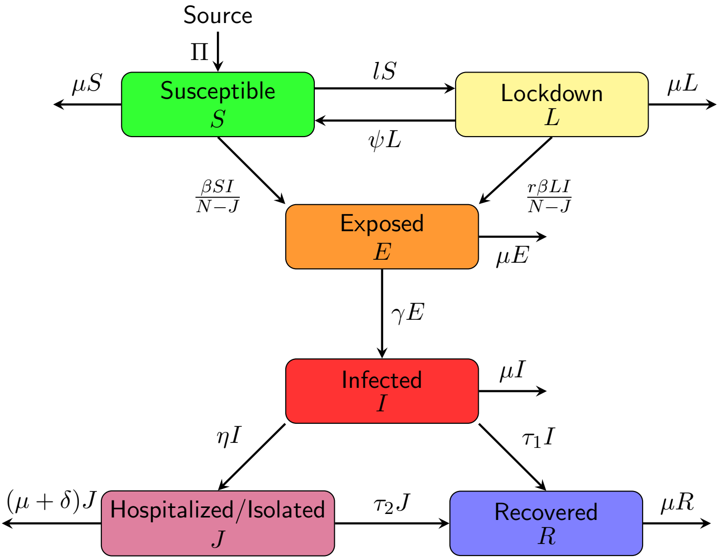
$$\frac{dE}{dt} = \frac{\beta S I}{N - J} + \frac{r\beta L I}{N - J} - (\gamma + \mu)E,$$

$$\frac{dI}{dt} = \gamma E - (\eta + \tau_1 + \mu)I,$$

$$\frac{dJ}{dt} = \eta I - (\tau_2 + \delta + \mu)J,$$

$$\frac{dR}{dt} = \tau_1 I + \tau_2 J - \mu R.$$

Flow Chart



Source
Π
Susceptible
S
μS
lS
ψL
Lockdown
L
μL
$\frac{\beta SI}{N-J}$
$\frac{r\beta LI}{N-J}$
Exposed
E
μE
γE
Infected
I
μI
ηI
$\tau_1 I$
$(\mu+\delta)J$
Hospitalized/Isolated
J
$\tau_2 J$
Recovered
R
μR

Well-posedness

The model is biologically well posed by Theorem 3.1, which guarantees two fundamental properties.

1. For $t > 0$, solutions with positive initial data remain positive.
2. The biologically feasible region Ω, a subset of six-dimensional Euclidean space is positively invariant and globally attracting.

$$\Omega = \left\{ (S, L, E, I, J, R) \in \mathbb{R}_+^6 : S + L + E + I + J + R \leq \frac{\Pi}{\mu} \right\}$$

What is $\mathcal{R}_0$?

1. $\mathcal{R}_0$ is defined as "the number of new infections produced by a typical infective individual in a population at a disease free equilibrium (DFE)".
2. More formally, it's the spectral radius of the next generation operator at disease free equilibrium (DFE).
3. The authors define $\mathcal{R}_0$ using the FV^{-1} approach described in class.

Calculating $\mathcal{R}_0$

- $\varepsilon_0 = \left(\frac{\Pi(\mu+\psi)}{\mu(\mu+\psi+l)}, \frac{\Pi l}{\mu(\mu+\psi+l)}, 0, 0, 0, 0\right)$ (Calculate DFE)
- $X = (E(t), I(t), J(t))$ (Identify the infected classes)
- $\dot{X} = \mathcal{F} - \mathcal{V}$ (Decompose $\dot{X}$)
- Linearize $\mathcal{F}$ and $\mathcal{V}$ at ε_0

$$F = \begin{pmatrix} 0 & \frac{\beta(\mu+\psi+rl)}{\mu+\psi+l} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V = \begin{pmatrix} \gamma+\mu & 0 & 0 \\ -\gamma & \eta+\tau_1+\mu & 0 \\ 0 & -\eta & \tau_2+\delta+\mu \end{pmatrix}$$

- $\mathcal{R}_0 = \rho(FV^{-1}) = \dfrac{\beta\gamma(\mu+\psi+rl)}{(\mu+\gamma)(\eta+\tau_1+\mu)(\mu+\psi+l)}$

Insights from $\mathcal{R}_0$

- What can we say about ε_0?

Lemma 3.1

The DFE is LAS whenever $\mathcal{R}_0 < 1$ and unstable whenever $\mathcal{R}_0 > 1$.

- How about endemic equilibria (EE)?

Theorem 3.2

Let P_1, P_2 and P_3 denote sets of certain parameter conditions. The model has

1. a unique EE if $P_1 \iff \mathcal{R}_0 > 1$,
2. a unique EE if P_2,
3. two EEs if P_3,
4. no EEs otherwise.

Insights from $\mathcal{R}_0$

Theorem 3.2

Let P_1, P_2 and P_3 denote sets of certain parameter conditions. The model has

1. a unique EE if $P_1 \iff \mathcal{R}_0 > 1$,
2. a unique EE if P_2,
3. two EEs if P_3,
4. no EEs otherwise.

- Take another look at case 3.
- What could having two EEs mean?
- To analyze this, we pick a 'nice' quantity to play with.
- Let $\varepsilon^* = (S^*, L^*, E^*, I^*, J^*, R^*)$ be any EE.
- Define $\lambda_h^* = \dfrac{\beta I^*}{N^* - J^*}$ (the 'force of infection')

Super Unfortunate Result

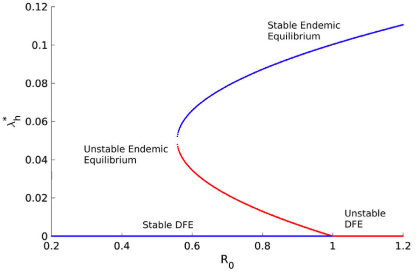


Fig. 2. Backward bifurcation diagram for the force of infection (λ_h^*) of the model (2.1). Using the parameter values: $\psi = 0.000246$, $\mu = 0.0049$, r=0.09, $\gamma = 0.0016$, $\eta = 0.0159$, β=5.905, $\tau_1 = 0.0101$, $\tau_2 = 0.0094$, $\delta = 0.0332$, l=0.09.

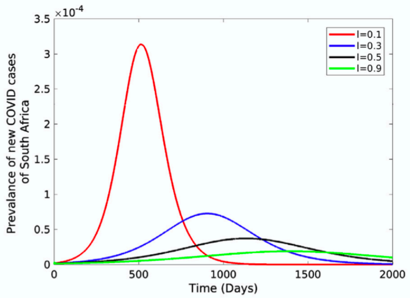
- (WOAH!) Stable DFE and stable EE can coexist with $\mathcal{R}_0 < 1$!
- Need to get $\mathcal{R}_0 < \mathcal{R}_0^c$ ($\mathcal{R}_0^c$ point of saddle node bifurcation).
- How?

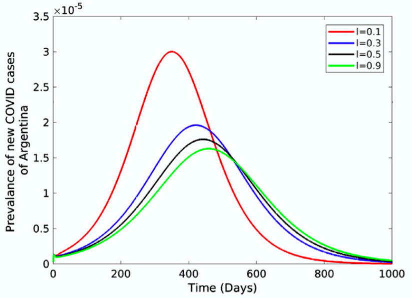
How to decrease $\mathcal{R}_0$?

- We start by highlighting an important result in the paper.

$$\frac{\partial \mathcal{R}_0}{\partial l} = -\frac{(1-r)\beta\gamma(\mu+\psi)}{(\gamma+\mu)(\eta+\tau_1+\mu)(\mu+\psi+l)^2} < 0$$

- Increases in lock down success (l) always reduce $\mathcal{R}_0$.
- Not particularly enlightening (as we defined l to have that effect), but still good to see that the model makes sense.





What else can be done?

- Aim for perfect lockdown efficacy ($r = 0$). Set $r = 0$ in ODEs.
- Obtain a reduced model M with $\mathcal{R}_0^* = \frac{\beta\gamma(\mu+\psi)}{(\mu+\gamma)(\eta+\tau_1+\mu)(\mu+\psi+1)}$.

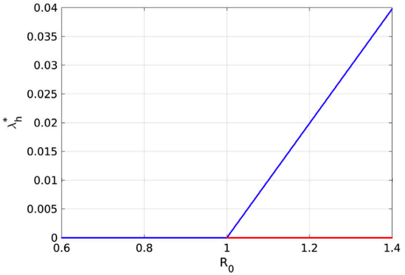


Fig. 3. Transcritical bifurcation for the force of infection (λ_h^*) of the model (2.1). Using the parameter values: $\psi = 0.000246$, $\mu = 0.0049$, r=0, $\gamma = 0.0016$, $\eta = 0.0159$, $\tau_1 = 0.0101$, β=0.5.905, $\tau_2 = 0.0094$, $\delta = 0.0332$, l=0.09.

Theorem 3.4

The DFE of M is GAS in Ω whenever $\mathcal{R}_0^* \leq \frac{\mu+\psi}{\mu+\psi+l} < 1$.

Recruitment into lockdown

- In model, recruited humans (Π) feed susceptible pool only.
- However Π is defined as the immigration or birth rate.
- Most countries during the COVID-19 crisis have travel advisories in place that restrict entirely, or severely limit the inflow of incoming travellers.
- If travellers are permitted into a country like Canada for instance, they're told to self isolate from the moment they step foot into the country to 14 days after that day.
- It would be more realistic to have the travel rate flow to the lock down population, or to split incoming travellers into a proportion that do self isolate and those that don't, and adjust the flow accordingly.

Final Takeaways

- Not always safe to strictly aim for $\mathcal{R}_0 < 1$,
- Bifurcation analysis is critical in understanding complex dynamics,
- Lockdowns **DO** help.