# 8 Space

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

# Mathematics 4MB3/6MB3
# Mathematical Biology

Instructor: David Earn

Lecture 8
Space
Tuesday 29 October 2024

# Announcments

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- **Midterm test:**
  - *Date:* Tuesday 12 November 2024
  - *Time:* 2:30pm–4:30pm
  - *Location:* in class, HH-102

  - Test structure will be discussed in class next week.

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- **Assignment 4** is due the day before the midterm.

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- **Assignment 4** is due the day before the midterm.
  - Make sure *you personally* can do the question on calculating $\mathcal{R}_0$ on this assignment *before* the midterm test.

# Spatial Epidemic Dynamics

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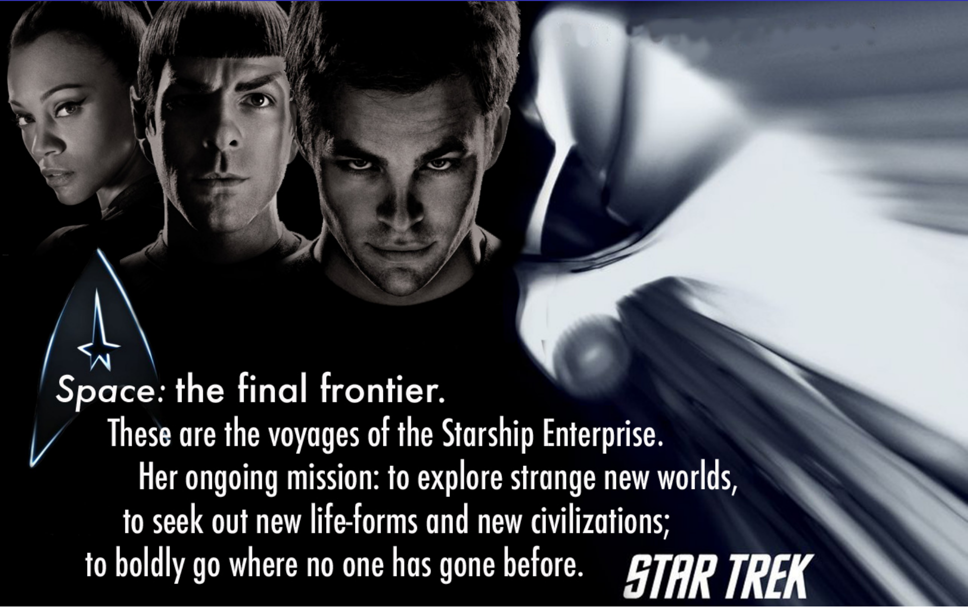

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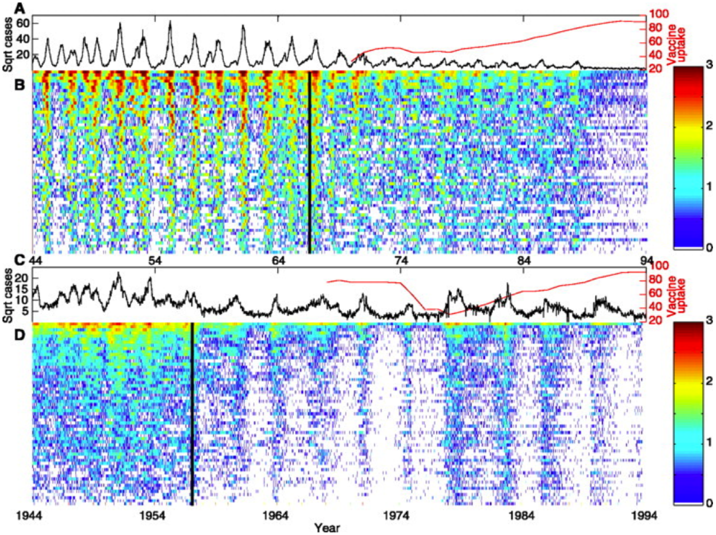
- Can we reduce the eradication threshold below $p_{\rm crit} = 1 - \frac{1}{\mathcal{R}_0}$?

# Measles and Whooping Cough in 60 UK cities

Measles

Whooping Cough




Rohani, Earn & Grenfell (1999) *Science* **286**, 968–971

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- But analytical theory of synchrony in a periodically forced system of differential equations is mathematically demanding...
- So let's consider a much simpler biological model...

# The Logistic Map

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- What kinds of dynamics are possible for the Logistic Map?

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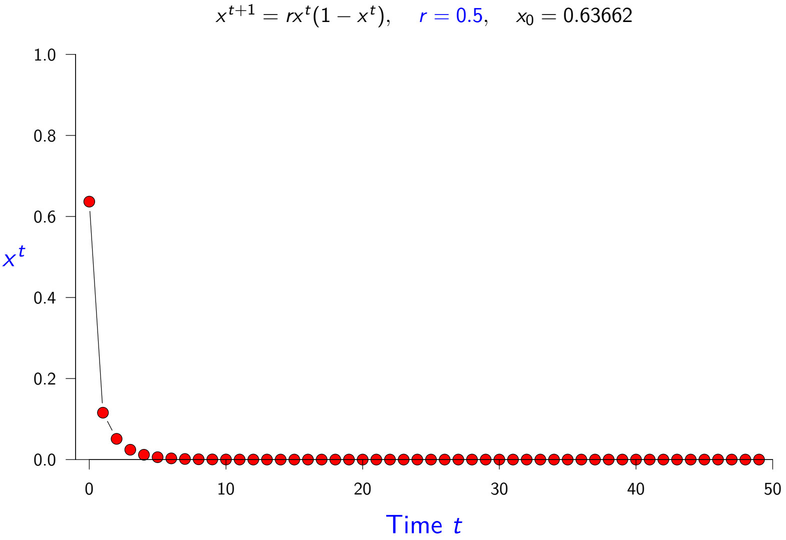

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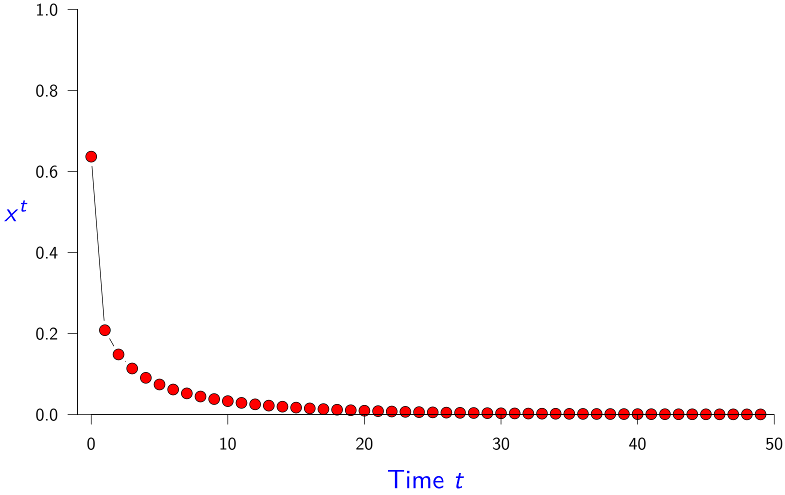

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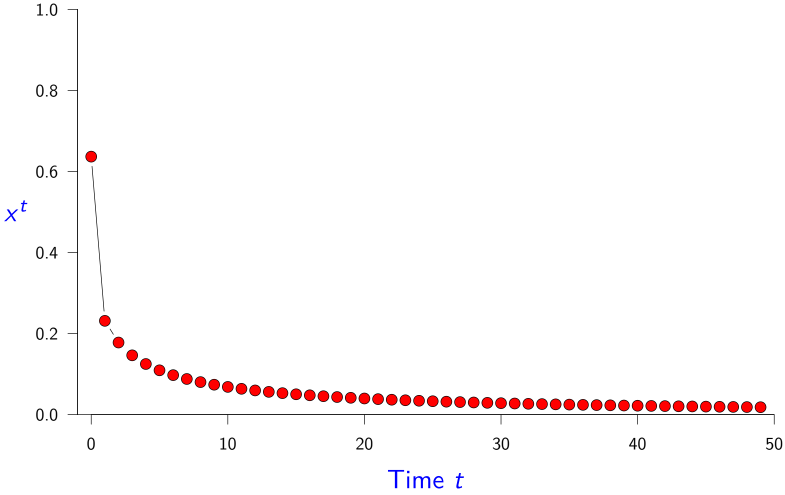

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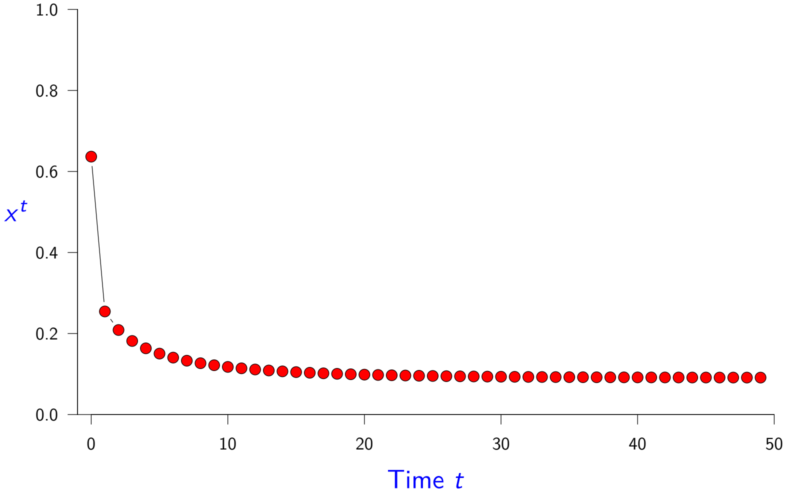

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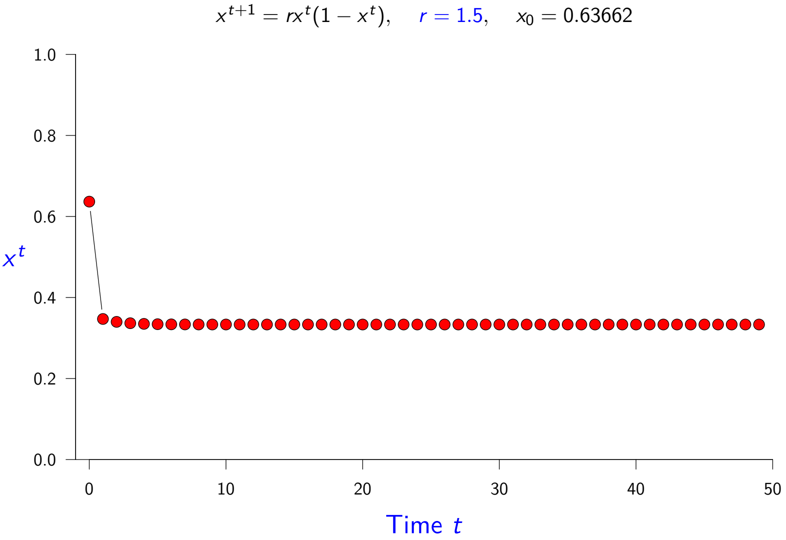

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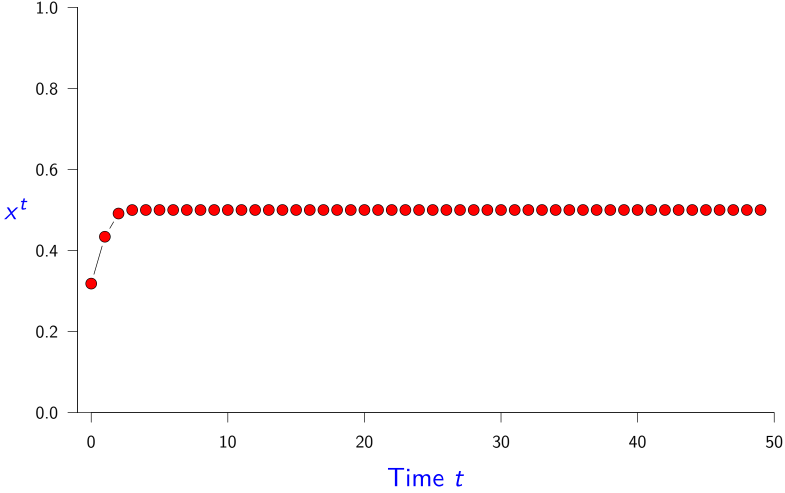

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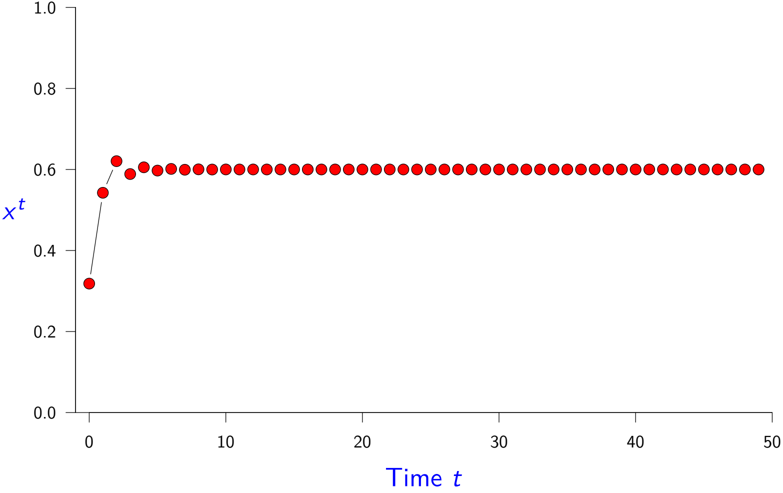

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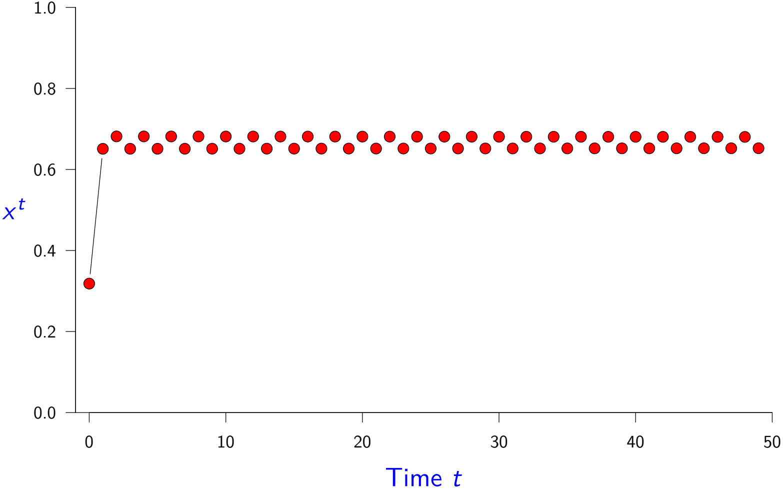

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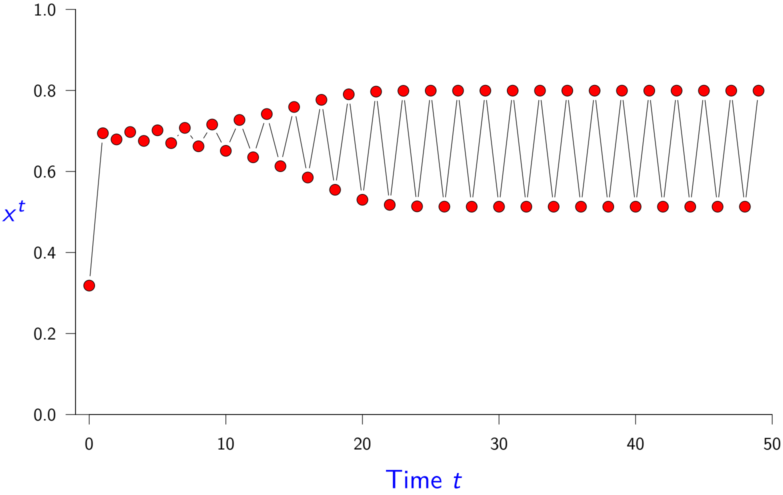

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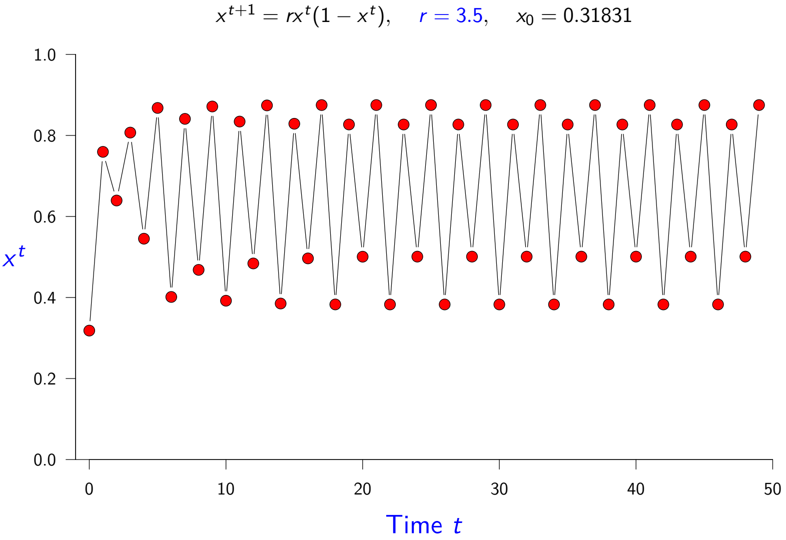

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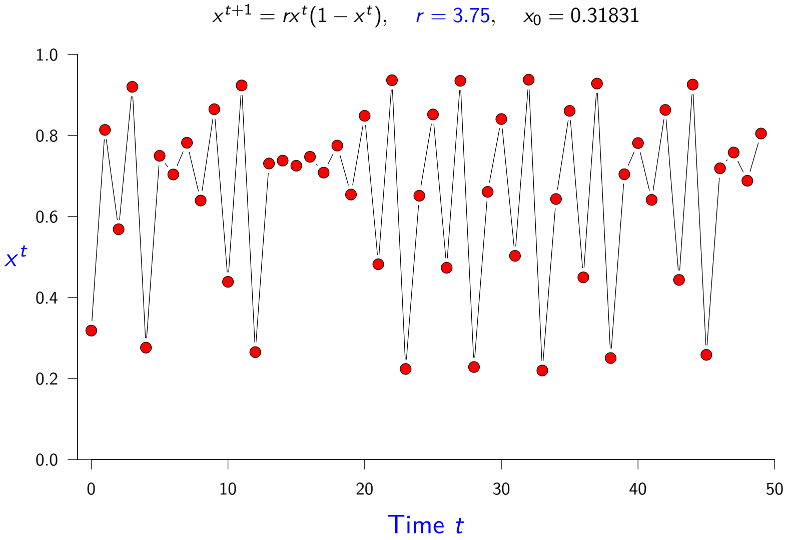

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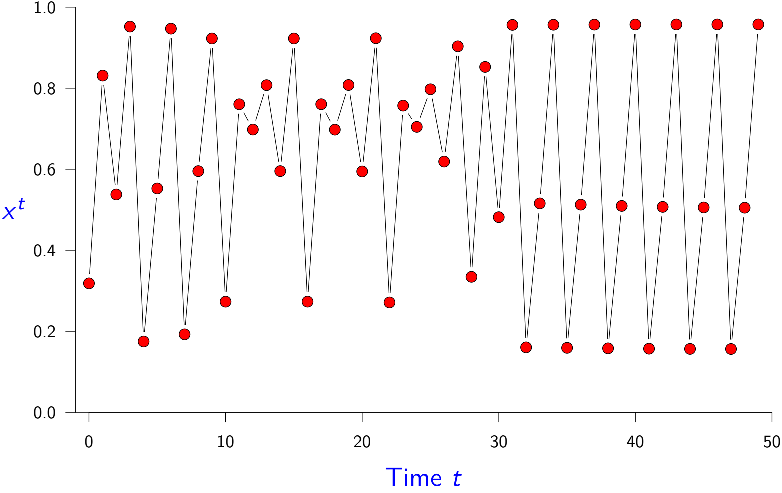

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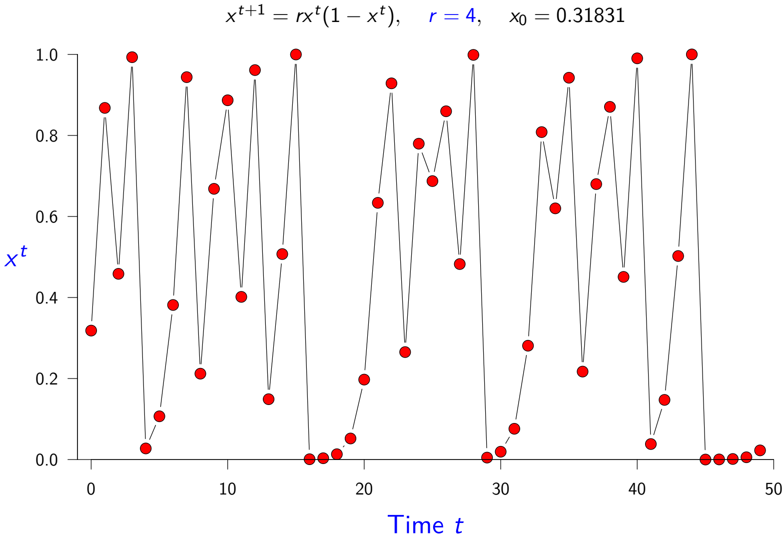

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  - Ignore transient behaviour: just show attractor.

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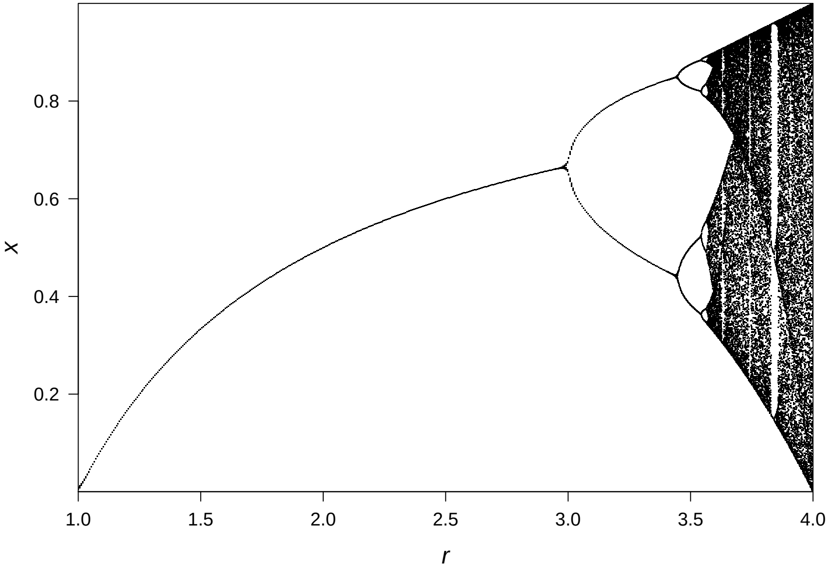
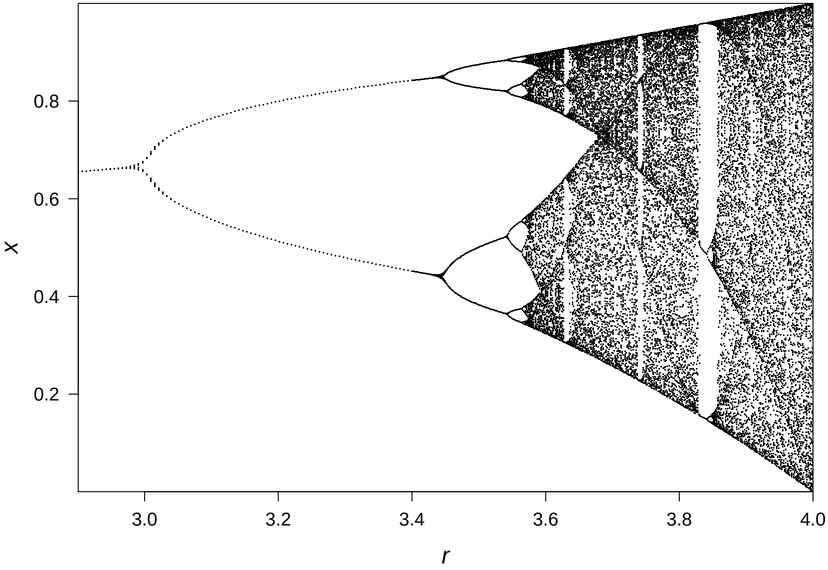


# Logistic Map, $F(x) = rx(1-x)$, $\quad 2.9 \le r \le 4$



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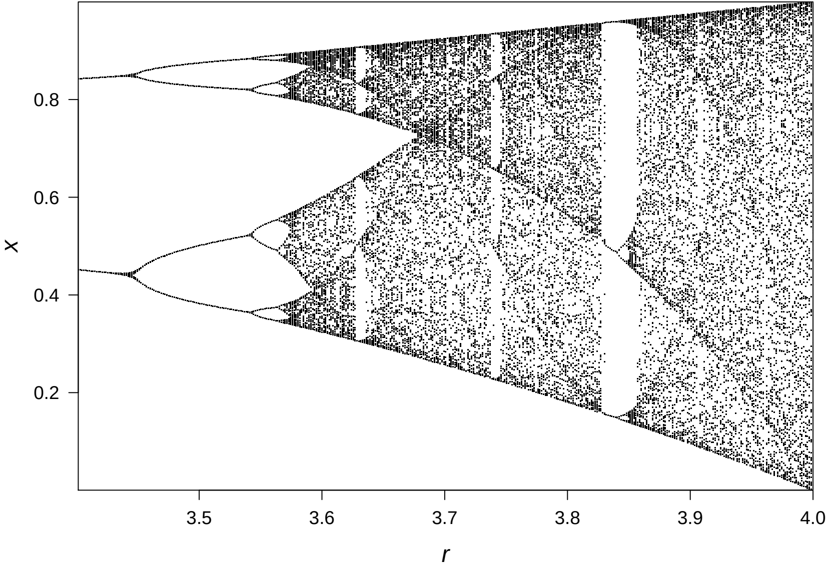

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- System is *coherent* (perfectly synchronous) if the state is the same in all patches
  *i.e.*, $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_n$

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- *Colour coding of matrix indices:*
  - row indices are red
  - column indices are cyan

# Basic properties of dispersal matrices $M = (m_{ij})$

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*so any coherent state can be written $x\boldsymbol{e}$, for some $x \in \mathbb{R}$.*

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# Back to Space and Synchrony

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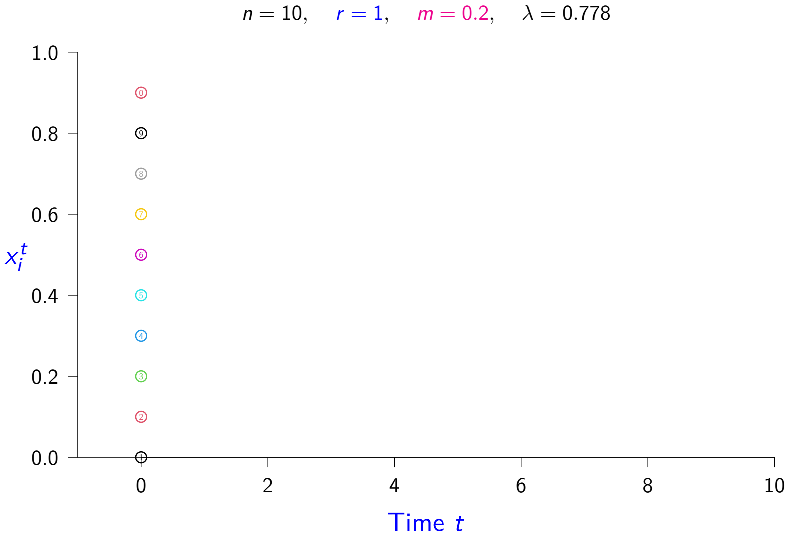

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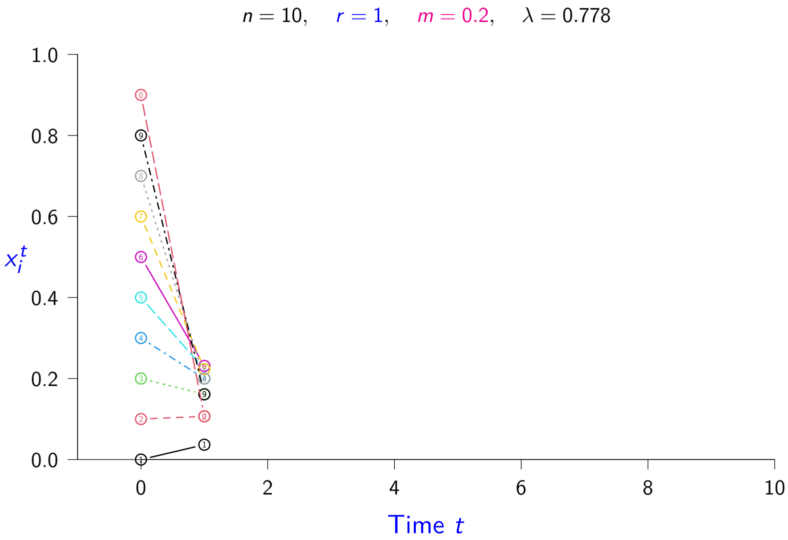

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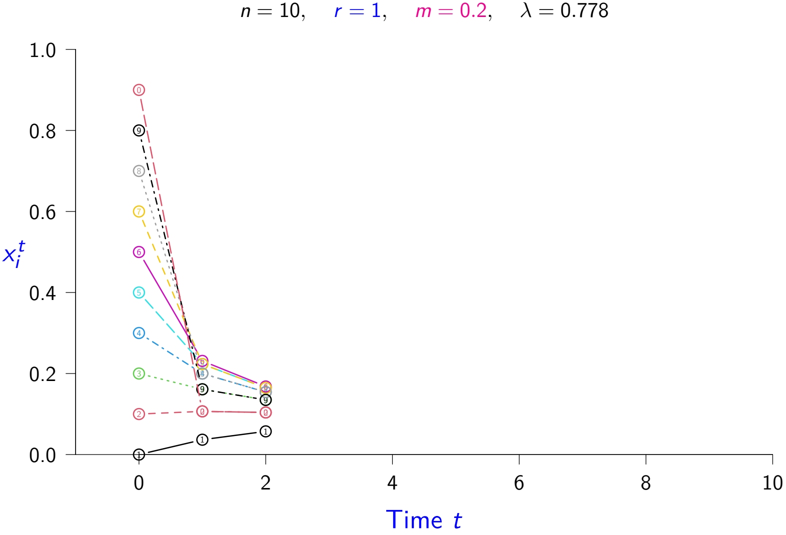

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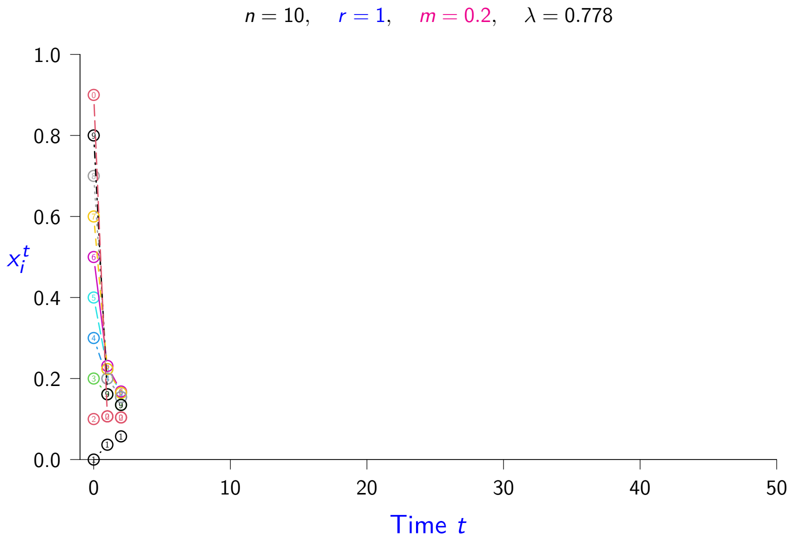

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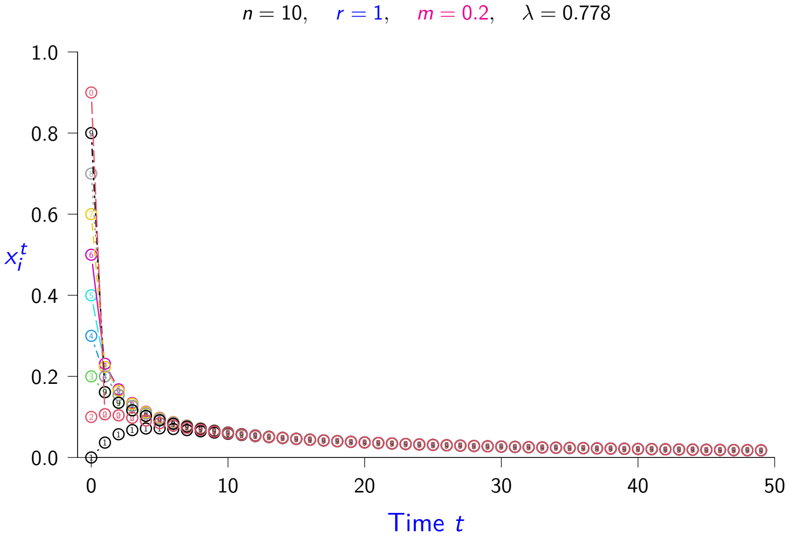

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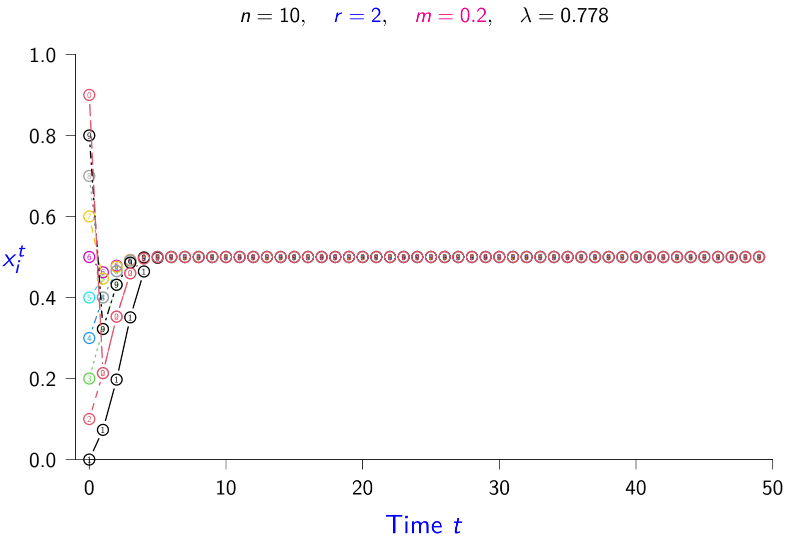

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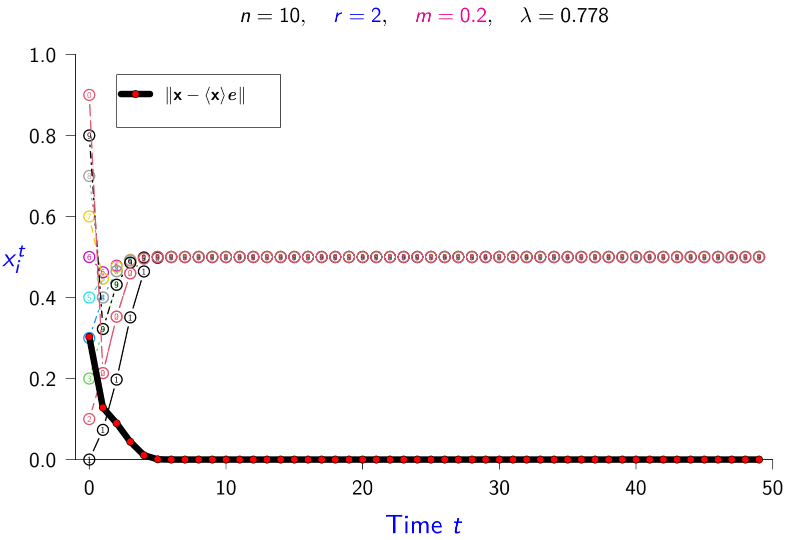

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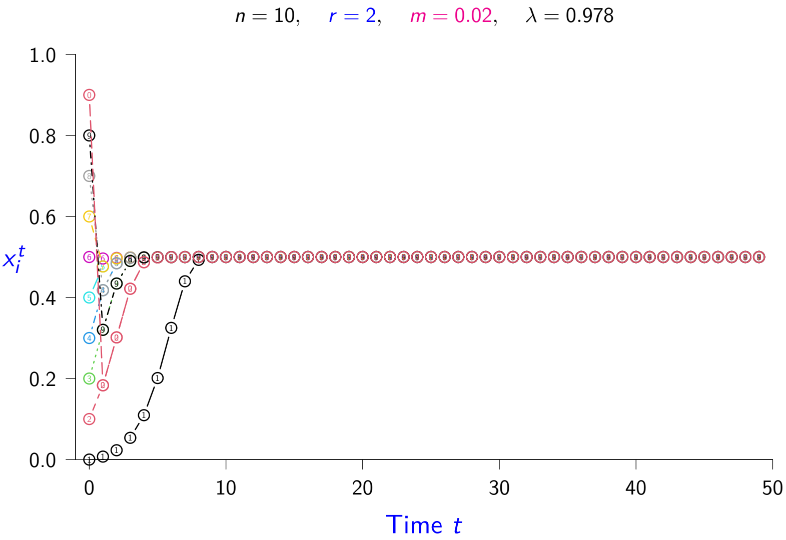

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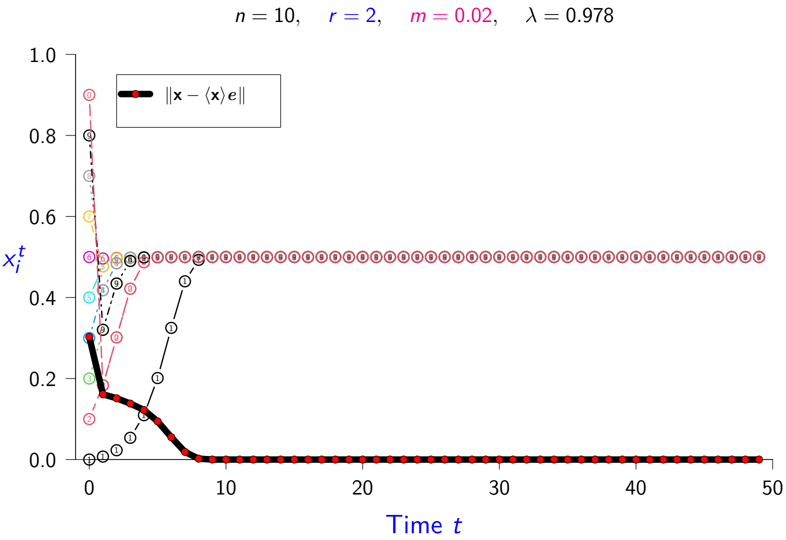

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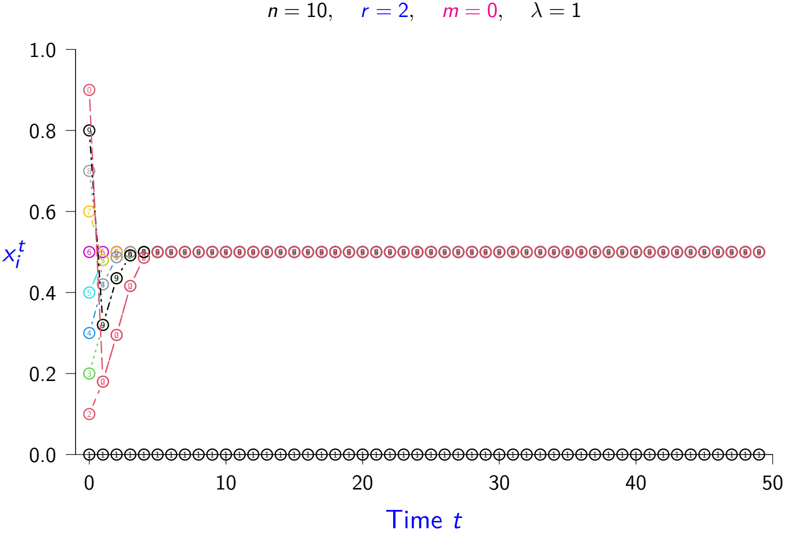

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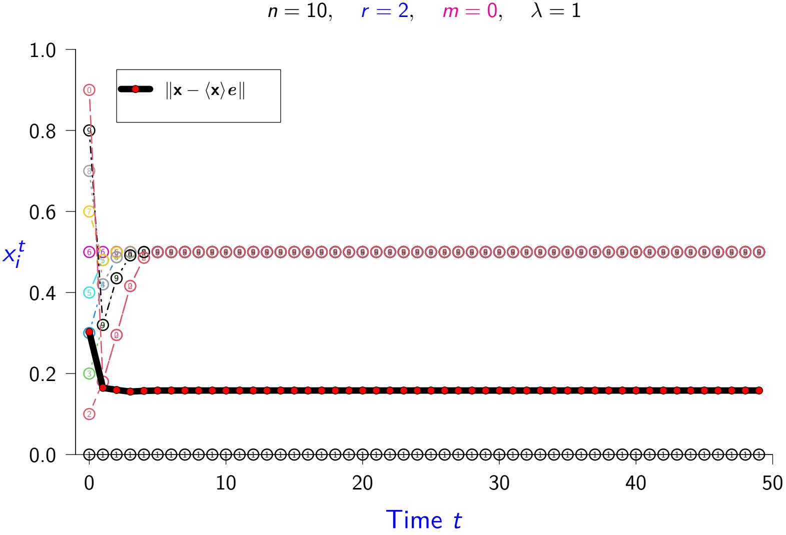

# Logistic Metapopulation Simulation ($r = 3.2,\ m = 0.2$)

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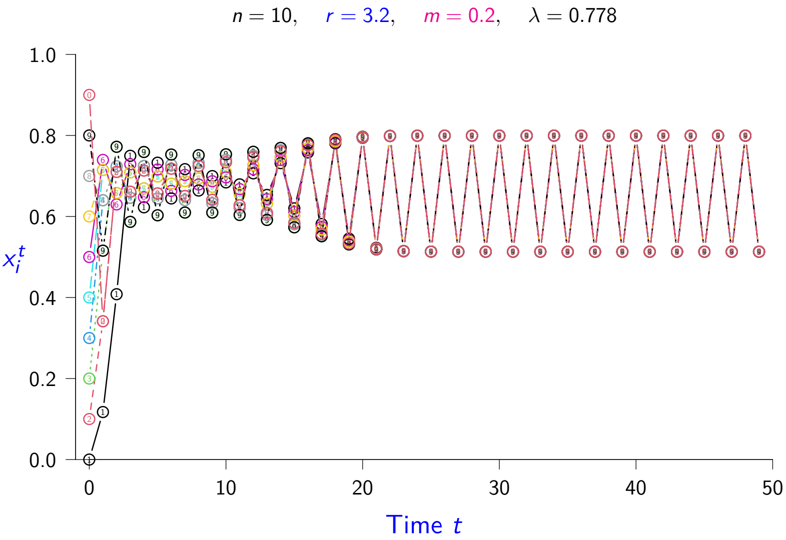
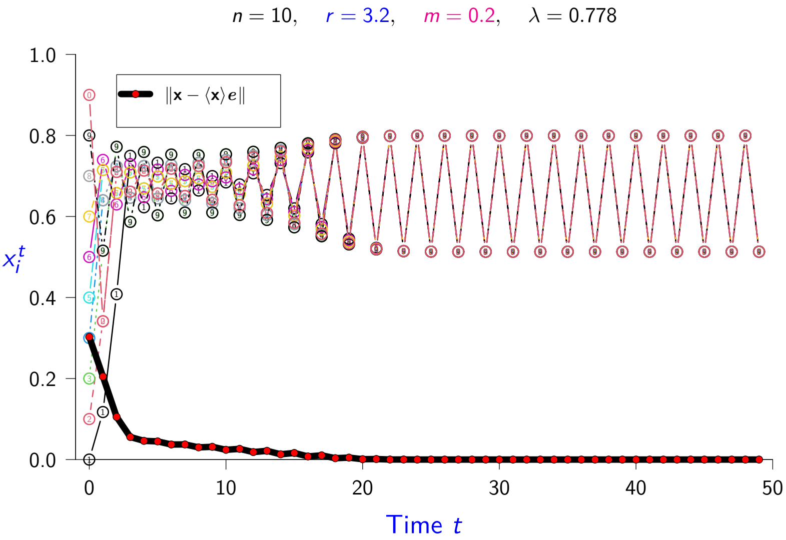


# Logistic Metapopulation Simulation ($r = 3.2$, $m = 0.2$)



# Logistic Metapopulation Simulation ($r = 3.5$, $m = 0.2$)

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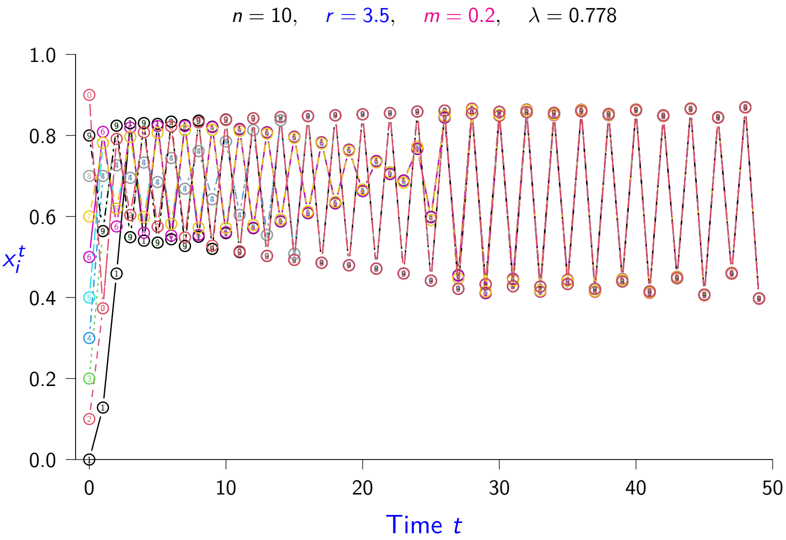
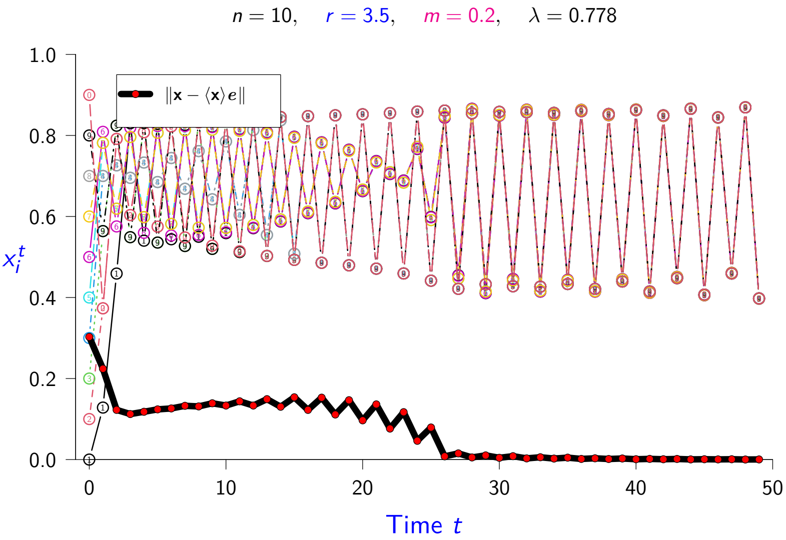


# Logistic Metapopulation Simulation ($r = 3.5,\ m = 0.2$)



# Logistic Metapopulation Simulation ($r = 3.75$, $m = 0.2$)

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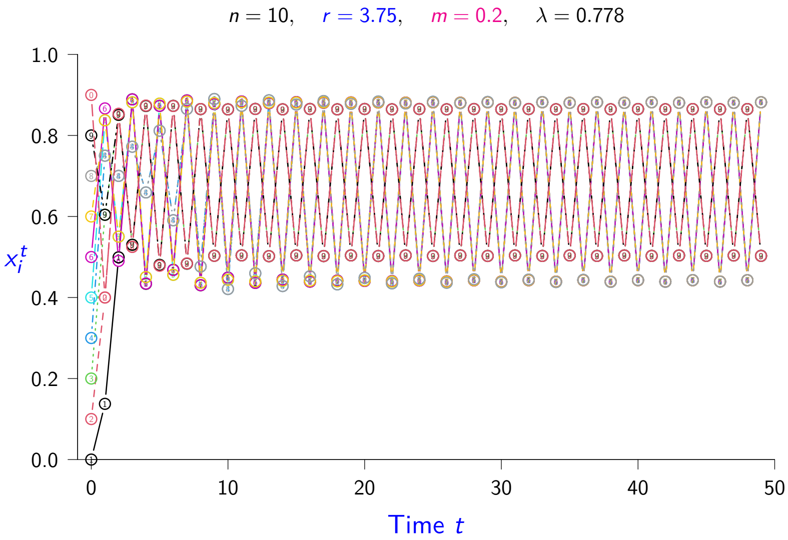

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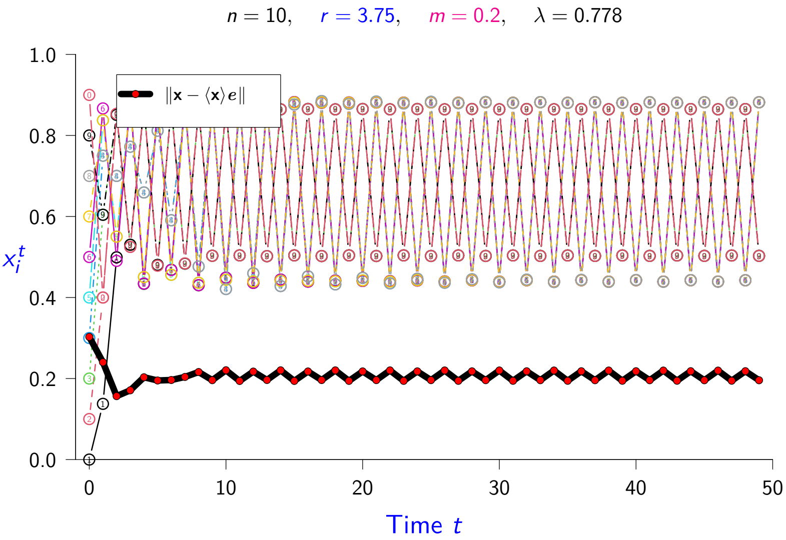

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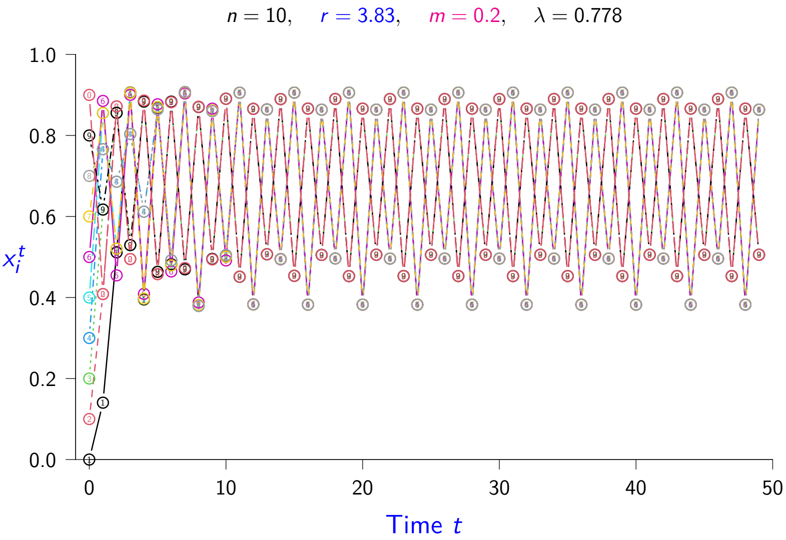

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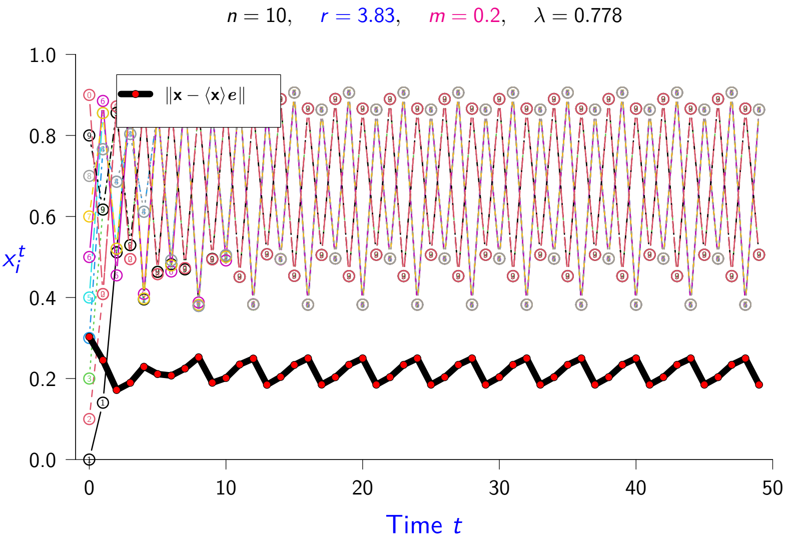

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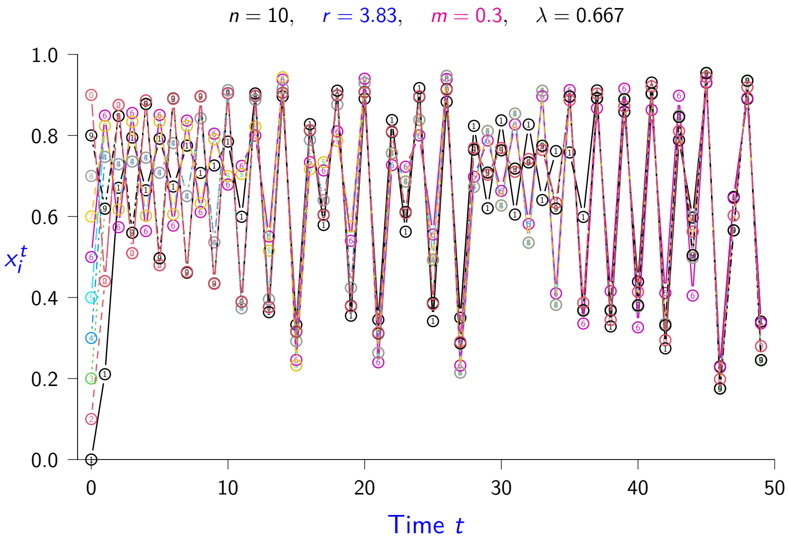

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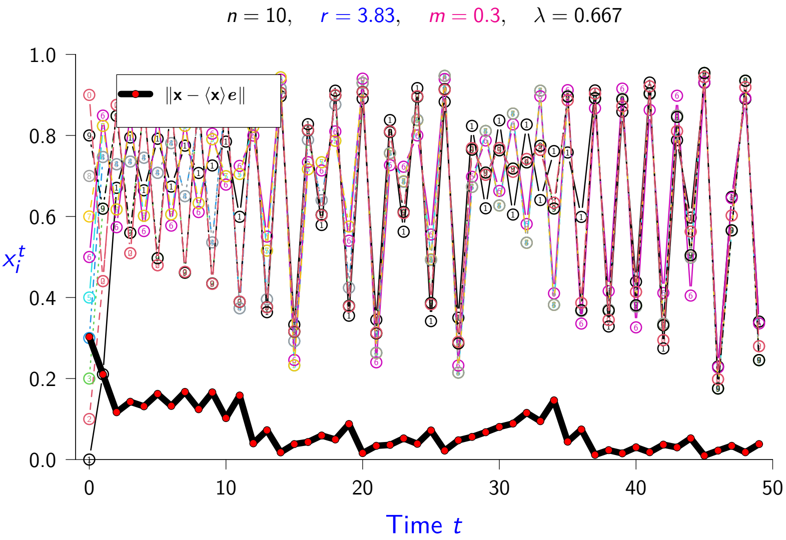

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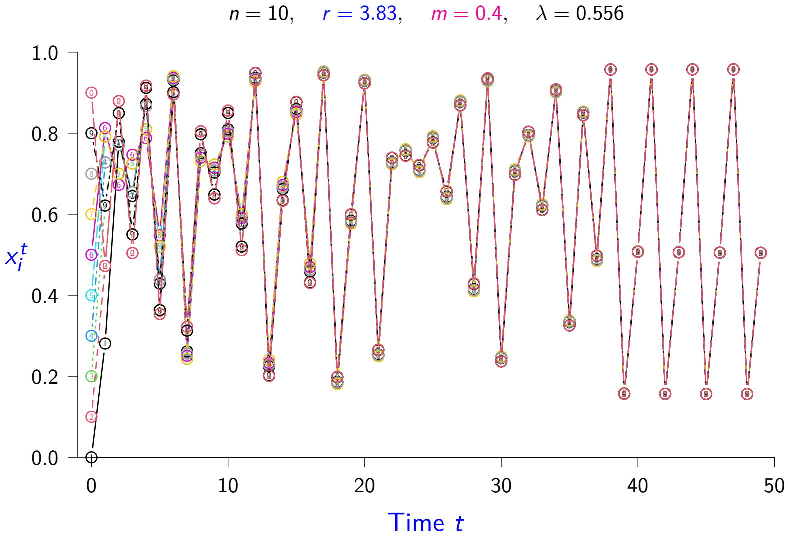

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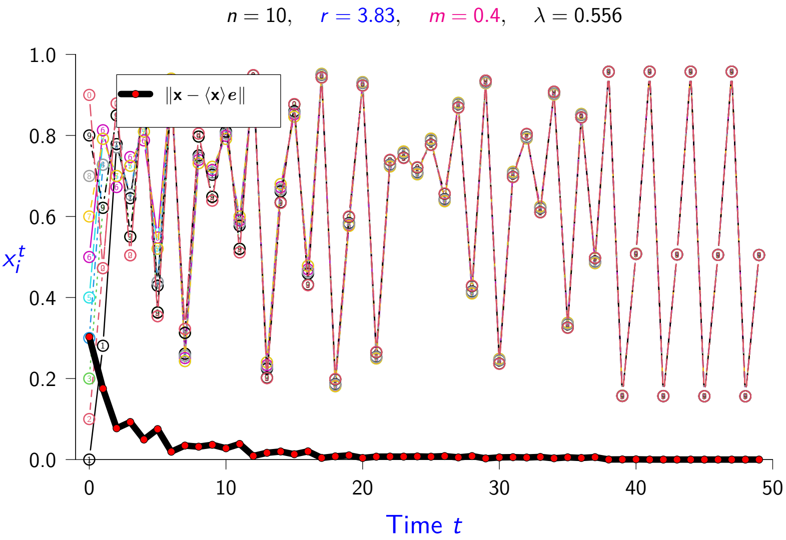

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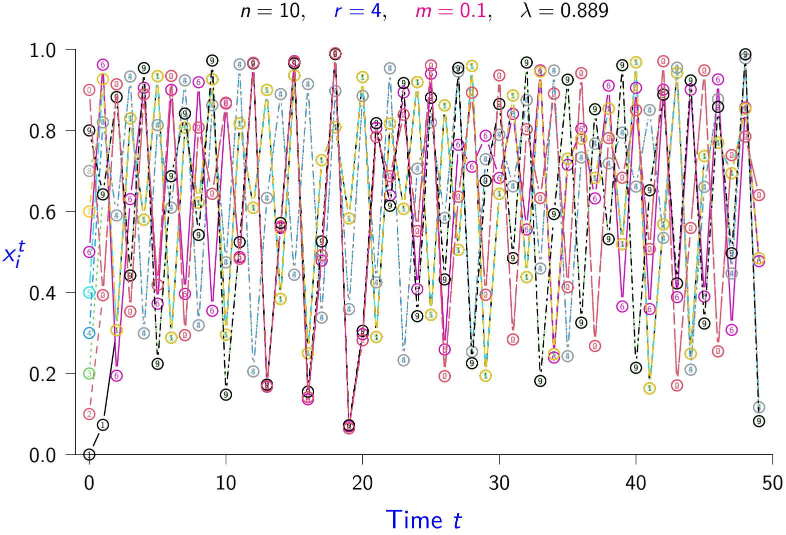

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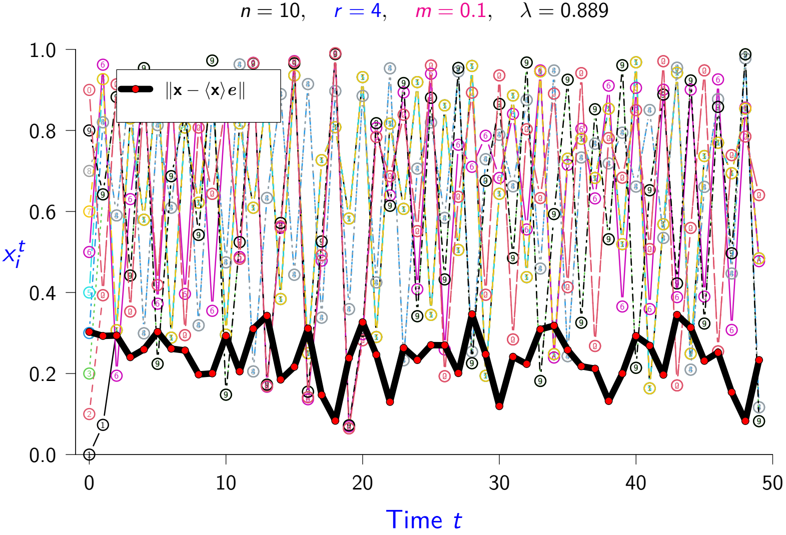

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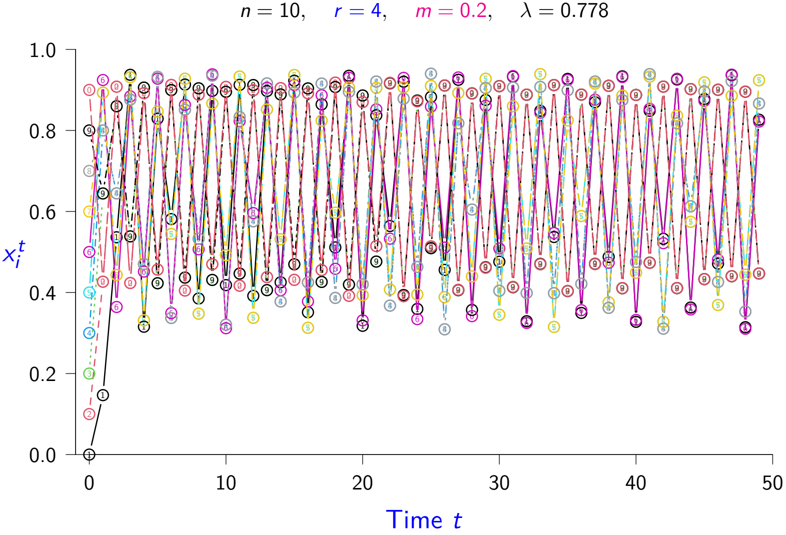

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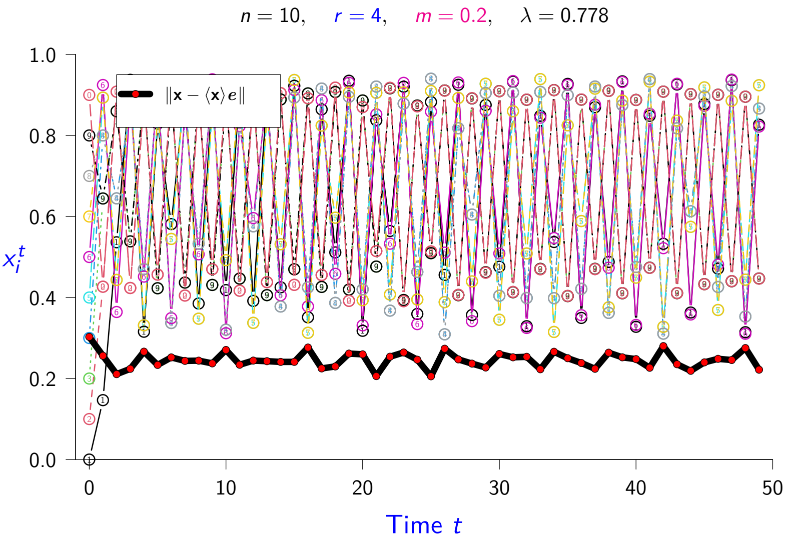

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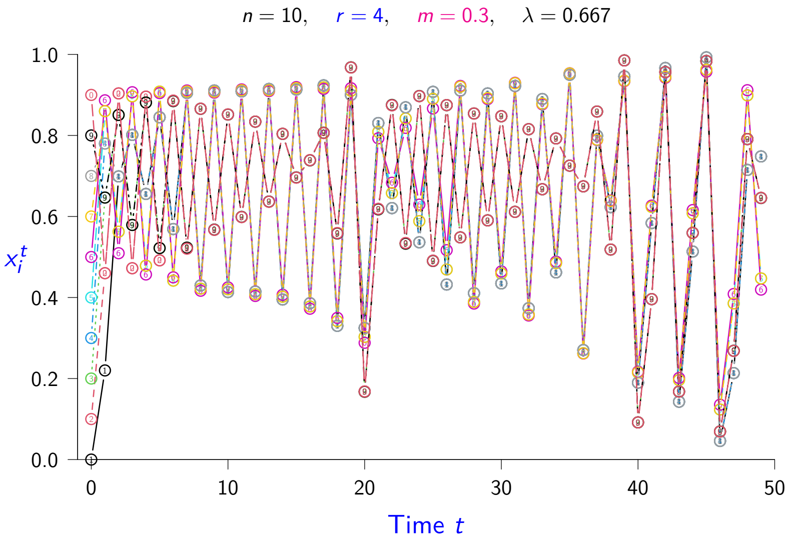

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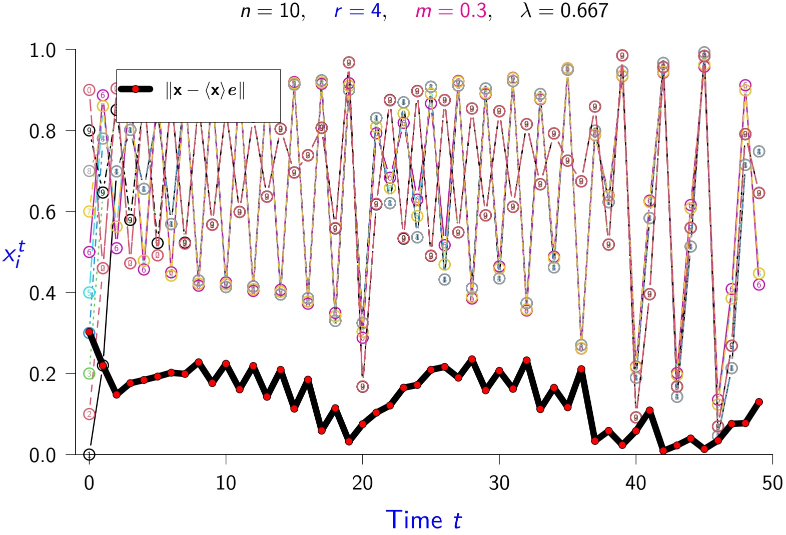

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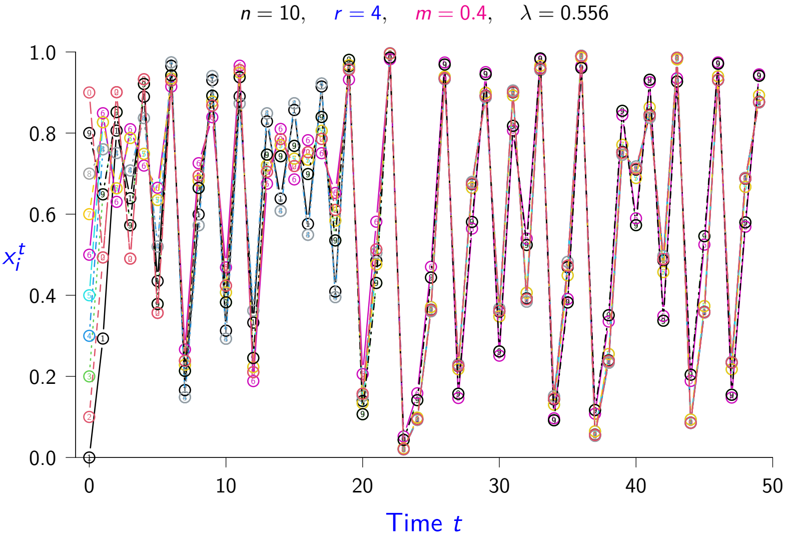

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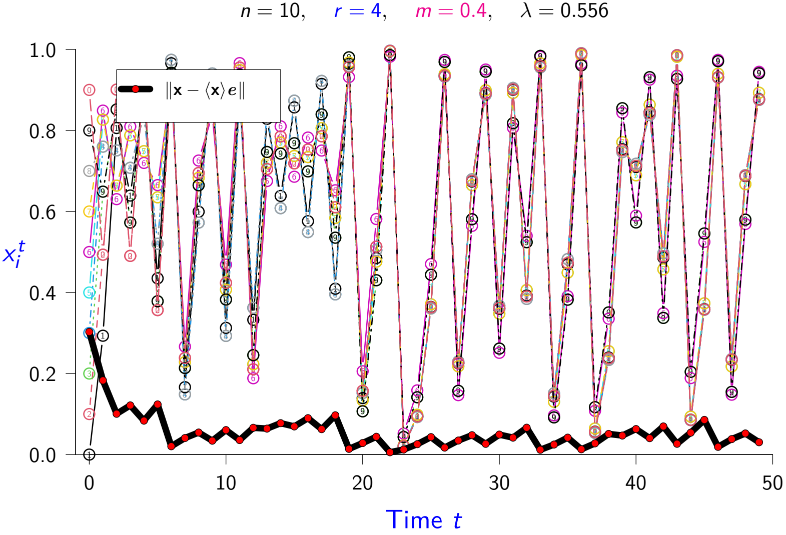

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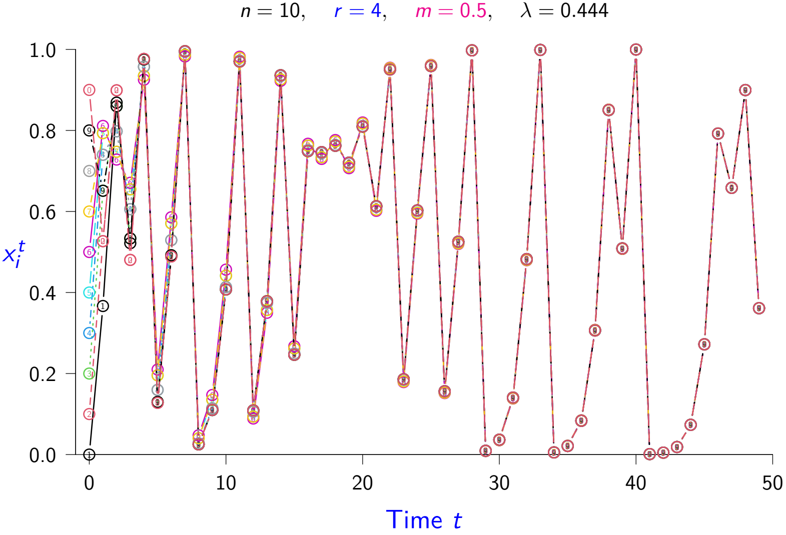

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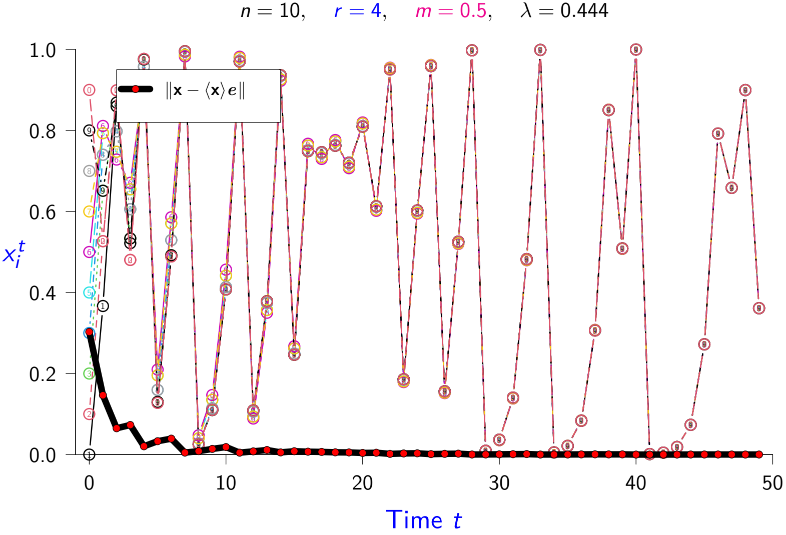

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- Logistic Metapopulation Simulations (10 patches)

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- Coherence is affected by magnitude $|\lambda|$ of *subdominant eigenvalue* $\lambda$.

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See Horn & Johnson (2013) *Matrix Analysis*, Corollary 8.1.30, p. 522. □

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But $\boldsymbol{e}$ is a positive vector, hence by the lemma on the previous slide, 1 is a dominant eigenvalue of M. □

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  This is called the maximum (Lyapunov) *characteristic exponent* of the single patch map.

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