McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 28

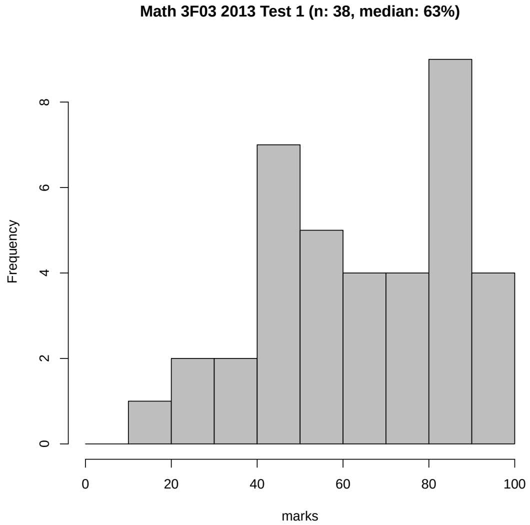
Lyapunov Functions

Wednesday 13 November 2013

Announcements

- **Assignment 4:**
  - Due this Friday 15 Nov 2013, 1:30pm.
- **Midterm Test #1:**
  - Marking complete.

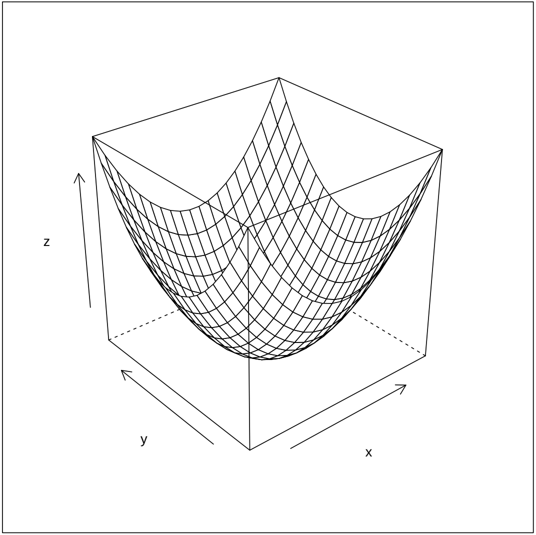
Test 1 Results



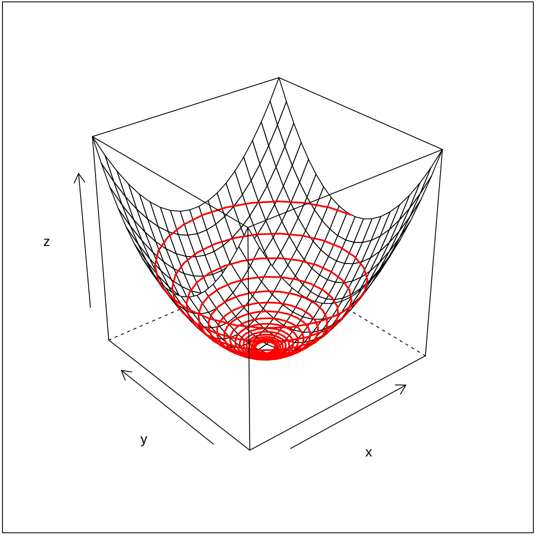
Establishing Stability

- If X_* is a hyperbolic equilibrium then linearize and use the linearization (Hartman-Grobman) theorem.
- If X_* is a non-hyperbolic equilibrium then linearization tells us nothing and we must use other methods to establish stability or instability.
- Even for hyperbolic X_*, if we care about the extent of the set of initial conditions that converge to X_* (the "basin of attraction of X_*") then linearization is not enough. Linearization yields only "local" information.

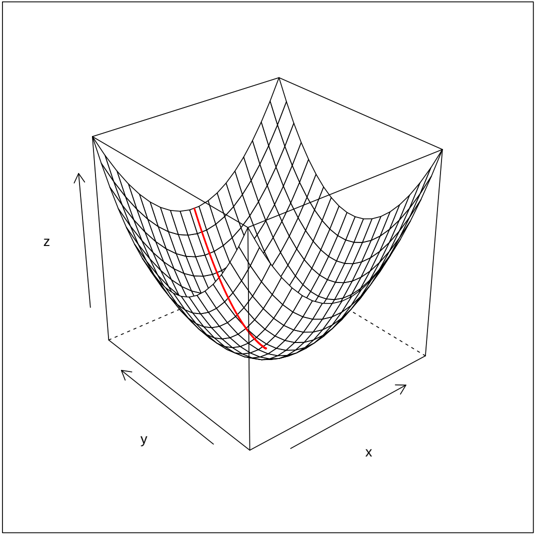
Lyapunov function idea: Surface $z = L(x, y)$



Trajectory: $(x(t), y(t), z(t))$, $z(t) = L(x(t), y(t))$



Trajectory: $(x(t), y(t), z(t))$, $z(t) = L(x(t), y(t))$



Establishing Stability: Lyapunov's Direct Method

Theorem (Lyapunov's Direct Method)

Consider an equilibrium X_* of $X' = F(X)$ and an open set $\mathcal{S}$ containing X_*. If $\exists$ a differentiable function $L : \mathcal{S} \to \mathbb{R}$ such that

(a) $L(X_*) = 0$ *and* $L(X) > 0 \quad \forall X \in \mathcal{S} \setminus \{X_*\}$
(L positive definite on $\mathcal{S}$)

(b) $\dot{L}(X) \leq 0 \quad \forall X \in \mathcal{S} \setminus \{X_*\}$ *($\dot{L}$ negative semi-definite on $\mathcal{S}$)*

then X_* is stable and L is called a **Lyapunov function***.*
If, in addition,

(c) $\dot{L}(X) < 0 \quad \forall X \in \mathcal{S} \setminus \{X_*\}$ *($\dot{L}$ negative definite on $\mathcal{S}$)*

then X_* is asymptotically stable and L is called a **strict Lyapunov function***.*

Discovering Lyapunov functions is an art!

Establishing Stability: Lyapunov's Direct Method

- Note that

$$\dot{L} = \frac{dL}{dt} = \nabla L \cdot X'$$

- So, condition for Lyapunov function is

$$L \geq 0 \quad \text{and} \quad \nabla L \cdot X' \leq 0 \qquad (X \neq X_*)$$

- Condition for strict Lyapunov function is

$$L \geq 0 \quad \text{and} \quad \nabla L \cdot X' < 0 \qquad (X \neq X_*)$$

Lyapunov function example

Example (1)

Find and classify all equilibria of the planar nonlinear system

$$x' = -3x^3 + 2xy^2$$
$$y' = -y^3$$

Solution.

$(x', y') = (0, 0) \iff (x, y) = (0, 0)$, so $\exists!$ equilibrium at $(0, 0)$.

$$DF_{(x,y)} = \begin{pmatrix} -9x^2 + 2y^2 & 4xy \\ 0 & -3y^2 \end{pmatrix}.$$

$$\therefore DF_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \text{non-hyperbolic.}$$

Cannot use linearization theorem. □

Lyapunov function example

$$x' = -3x^3 + 2xy^2$$
$$y' = -y^3$$

Solution (CONTINUED).

Try to find Lyapunov function $L(x,y)$ to prove $X_* = (0,0)$ is a stable equilibrium.

We need $L \geq 0$ and $\nabla L \cdot X' \leq 0$.

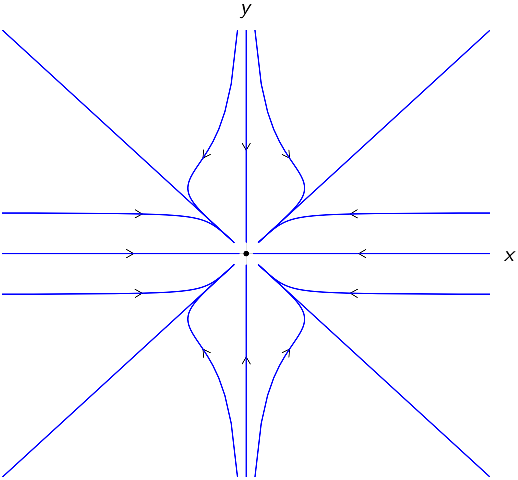
Try $L(x,y) = x^2 + y^2$. Then $L(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$, and

$$\begin{aligned}
\dot{L} &= \frac{dL}{dt} = \nabla L \cdot (x', y') \\
&= (2x, 2y) \cdot (-3x^3 + 2xy^2, -y^3) = -6x^4 + 4x^2y^2 - 2y^4 \\
&= -2\Big(3x^4 - 2x^2y^2 + y^4\Big) = -2\Big(2x^4 + (x^4 - 2x^2y^2 + y^4)\Big) \\
&= -2\Big(2x^4 + (x^2 - y^2)^2\Big) \\
&< 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{X_*\} \implies (0,0) \text{ is asymptotically stable.}
\end{aligned}$$

□

Lyapunov function example

$$x' = -3x^3 + 2xy^2$$
$$y' = -y^3$$



Lyapunov function hint

- The example just considered was contrived... Lyapunov functions are not usually perfect circular parabaloids!
- But, functions with quasi-elliptical level sets often work as Lyapunov functions, *e.g.*,

$$ax^2 + by^2$$
$$ax^4 + by^2$$
$$ax^2 + by^4$$
$$ax^4 + by^4$$
$$\vdots$$

- Often useful to try a function of the form

$$L(x, y) = ax^n + by^m$$

and then choose a, b, n, m so L has required properties.

Lyapunov function example

Example (2)

Find and determine the stability of all equilibria of

$$x' = -2y^3$$
$$y' = x - 3y^3$$

Solution.

$(x', y') = (0,0) \iff (x,y) = (0,0)$, so $\exists!$ equilibrium at $(0,0)$.

$DF_{(x,y)} = \begin{pmatrix} 0 & -6y^2 \\ 1 & -9y^2 \end{pmatrix}$.

$\therefore DF_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \implies$ non-hyperbolic.

Cannot use linearization theorem. □

Lyapunov function example

$$x' = -2y^3$$
$$y' = x - 3y^3$$

Solution (CONTINUED).

We need $L \geq 0$ and $\nabla L \cdot X' \leq 0$. Try $L(x,y) = ax^2 + by^4$, with $a, b \geq 0$. Then $L(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$, and

$$\dot{L} = \frac{dL}{dt} = \nabla L \cdot (x', y') = (2ax, 4by^3) \cdot (-2y^3, x - 3y^3)$$
$$= -4axy^3 + 4bxy^3 - 12by^6 = 4(b-a)xy^3 - 12by^6$$

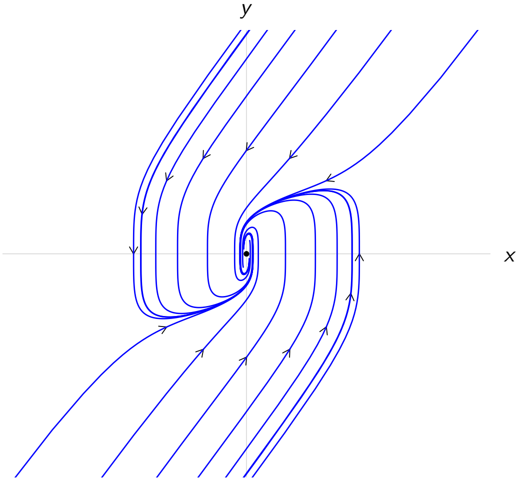
Choose $a = b > 0$ to eliminate first term and ensure second term ≤ 0. *e.g.*, $a = b = 1$ yields

$$\nabla L \cdot X' = -12y^6 \leq 0 \; \forall (x,y) \in \mathbb{R}^2 \setminus \{X_*\} \implies (0,0) \text{ is stable.}$$

Note that we CANNOT conclude that X_* is asymptotically stable: Whole x-axis ($y = 0$) yields $\nabla L \cdot X' = 0$. □

Lyapunov function example

$$x' = -2y^3$$
$$y' = x - 3y^3$$



Using Lyapunov functions to prove instability

Theorem (Lyapunov's instability theorem)

Consider an equilibrium X_* of $X' = F(X)$ and suppose $\exists$ differentiable function $L : B \to \mathbb{R}$, where B is an open ball in $\mathbb{R}^n$, such that

(a) $L(X_*) = 0$,

(b) *for any ball* $B_1 \subset B$, $\exists X_1 \in B_1$ *such that* $L(X_1) > 0$,

(c) $\dot{L}(X) > 0 \quad \forall X \in B \setminus \{X_*\}$ *($\dot{L}$ positive definite on B).*

Then X_* is unstable.

Example (3)

$$x' = -y^3$$
$$y' = -x^3$$

Lyapunov instability example

$$x' = -y^3$$
$$y' = -x^3$$

Solution.

$(x', y') = (0,0) \iff (x,y) = (0,0)$, so $\exists!$ equilibrium at $(0,0)$.

$$DF_{(x,y)} = \begin{pmatrix} 0 & -3y^2 \\ -3x^2 & 0 \end{pmatrix}.$$

$$\therefore DF_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \text{non-hyperbolic.}$$

Cannot use linearization theorem.

Look for a function $L(x,y)$ with a gradient ∇L that "dots to positive" with the vector field (x', y').

Consider $L(x,y) = -xy$.

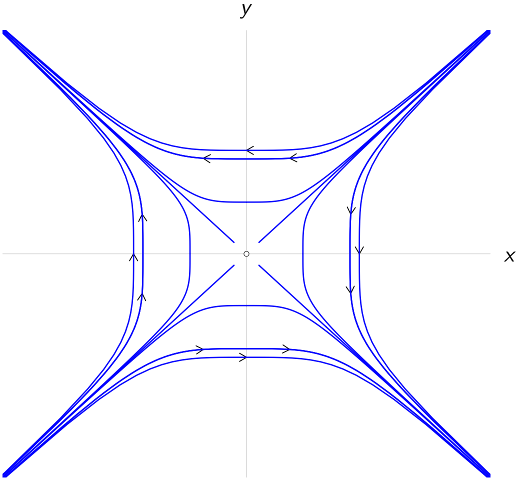
Then $L(0,0) = 0$ and $L(x,y) > 0$ in 2nd and 4th quadrants.

$$\begin{aligned} \dot{L}(X) &= \nabla L \cdot X' = (-y, -x) \cdot (-y^3, -x^3) \\ &= y^4 + x^4 > 0 \quad \text{except at origin.} \end{aligned}$$

$\therefore X_* = (0,0)$ is unstable. □

Lyapunov instability example

$$x' = -y^3$$
$$y' = -x^3$$



Local vs Global Stability

- We have talked only about **local stability** of equilibria (and other limit sets such as periodic orbits).
- But Lyapunov's method shows that all states in a given *ball* stay near or converge to X_*.
- If we can construct a strict Lyapunov function that works on *all* of $\mathbb{R}^n$ then *all* states will converge to X_* as $t \to \infty$.
- We then say X_* is **globally asymptotically stable** (**GAS**).
- Lyapunov functions are useful for proving GAS even if X_* can be shown to be **locally asymptotically stable** (**LAS**) via linearization.

Non-equilibrium Limit Sets

- Lyapunov's method can be used to establish stability (or instability) of other types of limit sets (*e.g.*, periodic orbits).
- As with equilibria, the challenge is to find L.
- There is no general way known to discover Lyapunov functions.