McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3F03
Advanced Differential Equations

Instructor: David Earn

Lecture 27
Notions of Stability
Monday 11 November 2013

Announcements

- **Assignment 4:**
  - Due this Friday 15 Nov 2013, 1:30pm.
- **Midterm Test #1:**
  - Marking still in progress.

(Lyapunov) Stability

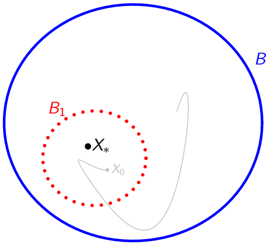
Intuition:
Equilibrium X_* is stable if nearby solutions stay nearby for all time.

Definition (Stable)

An equilibrium X_* is **stable** if, for any open ball B containing X_*, $\exists B_1 \subset B$ such that:

(i) $X_* \in B_1$,

(ii) $\forall X_0 \in B_1$, **if** $X(0) = X_0$ **then** $X(t) \in B \quad \forall t > 0$.



In words: Given any open ball (say B) containing X_*, there is another open ball (say B_1) within B and also containing X_*, such that any solution $X(t)$ that starts in B_1 stays forever within B.

Asymptotic Stability

Intuition:

Stable AND nearby solutions actually converge onto X_*.

Definition (Asymptotically Stable)

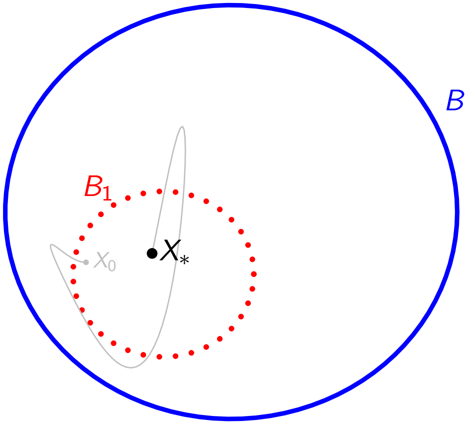
An equilibrium X_* is **asymptotically stable** if, for any open ball B containing X_*, $\exists B_1 \subset B$ such that:

(i) $X_* \in B_1$,

(ii) $\forall X_0 \in B_1, \quad X(0) = X_0 \implies X(t) \in B \quad \forall t > 0$,

(iii) $\forall X_0 \in B_1, \quad X(0) = X_0 \implies \lim_{t\to\infty} X(t) = X_*$.

Asymptotic Stability



Instability

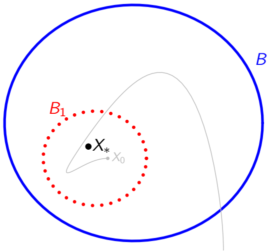
Intuition:
NOT stable, *i.e.*, no matter how close you get to X_*, there are solutions that eventually go "far away" from X_*.

Definition (Unstable)

An equilibrium X_* is **unstable** if $\exists$ an open ball B such that

(i) $X_* \in B$,

(ii) $\forall$ open balls $B_1 \subset B$, $\exists X_0 \in B_1$ and $t > 0$ such that $X(0) = X_0$ and $X(t) \notin B$.



In words: There is *some* open ball (say B) containing X_* such that *for any* open ball (say B_1) within B and containing X_*, there is *some* point $X_0 \in B_1$ such that the solution $X(t)$ that starts at X_0 eventually escapes from B.

Stability examples

Example (Stable equilibrium)

- Sink.
- Centre.
- Attracting line.
- $F(X) \equiv 0$ (all points fixed).

Example (Asymptotically stable equilibrium)

- Sink.

Stability examples

Example (Stable but not asymptotically stable)

- Centre.
- Attracting line.
- $F(X) \equiv 0$ (all points fixed).

Example (Unstable equilibrium)

- Source.
- Saddle.
- Repelling line.

Stability examples

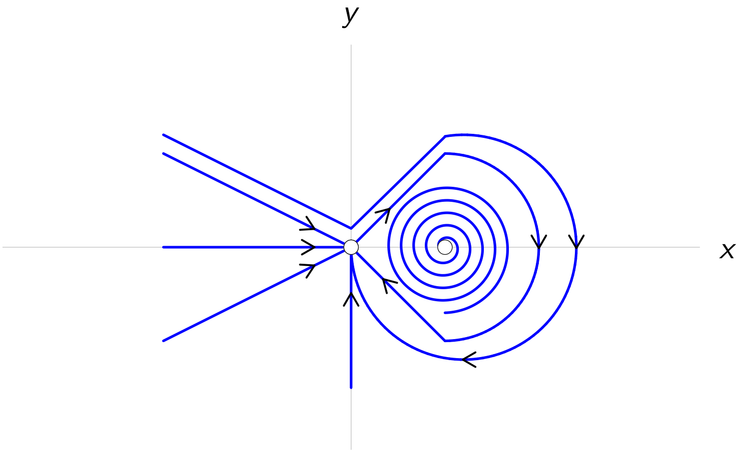
Why did we define asymptotically stable to be "stable AND converges"?

Does convergence imply (Lyapunov) stability?

Example (Asymptotic approach to equilibrium but not stable)

Is this actually possible?
If so, draw a picture that illustrates how.
If not, prove it (*i.e.*, prove that asymptotic convergence to an equilibrium implies stability in the sense of Lyapunov).

Example: convergent but unstable equilibrium



Extra Credit Problem

Construct an explicit nonlinear system $X' = F(X)$, $F \in C^1(\mathbb{R}^2)$, with a phase portrait that is topologically conjugate to the example on the previous slide.

Prove that your example has the desired properties, *i.e.*, that F is continuously differentiable and that the equilibrium at the origin is convergent but unstable.

Stability of more complicated invariant sets

The notions of stability and instability are also applicable to many other invariant sets, not just equilibria.

Example (Stable but not asymptotically stable invariant set)

Any periodic orbit in a centre
(not just the equilibrium).

Establishing Stability

- If X_* is a hyperbolic equilibrium then linearize and use the linearization (Hartman-Grobman) theorem.
- If X_* is a non-hyperbolic equilibrium then linearization tells us nothing and we must use other methods to establish stability or instability.
- Even for hyperbolic X_*, if we care about the extent of the set of initial conditions that converge to X_* (the "basin of attraction of X_*") then linearization is not enough. Linearization yields only "local" information.

Establishing Stability: Lyapunov's Direct Method

Theorem (Lyapunov's Direct Method)

Consider an equilibrium X_* of $X' = F(X)$ and an open set $\mathcal{S}$ containing X_*. If $\exists$ a differentiable function $L : \mathcal{S} \to \mathbb{R}$ such that

(a) $L(X_*) = 0$ *and* $L(X) > 0 \quad \forall X \in \mathcal{S} \backslash \{X_*\}$
(L positive definite on $\mathcal{S}$)

(b) $\dot{L}(X) \leq 0 \quad \forall X \in \mathcal{S} \backslash \{X_*\}$ *($\dot{L}$ negative semi-definite on $\mathcal{S}$)*

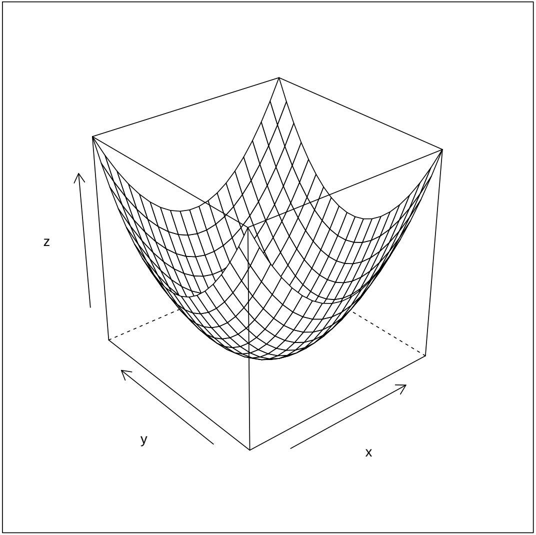
then X_* is stable and L is called a **Lyapunov function***.*
If, in addition,

(c) $\dot{L}(X) < 0 \quad \forall X \in \mathcal{S} \backslash \{X_*\}$ *($\dot{L}$ negative definite on $\mathcal{S}$)*

then X_* is asymptotically stable and L is called a **strict Lyapunov function***.*

Discovering Lyapunov functions is an art!

Lyapunov function idea: Surface $z = L(x, y)$



Trajectory: $(x(t), y(t), z(t))$, $z(t) = L(x(t), y(t))$