19 Space

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 4MB3/6MB3
Mathematical Biology

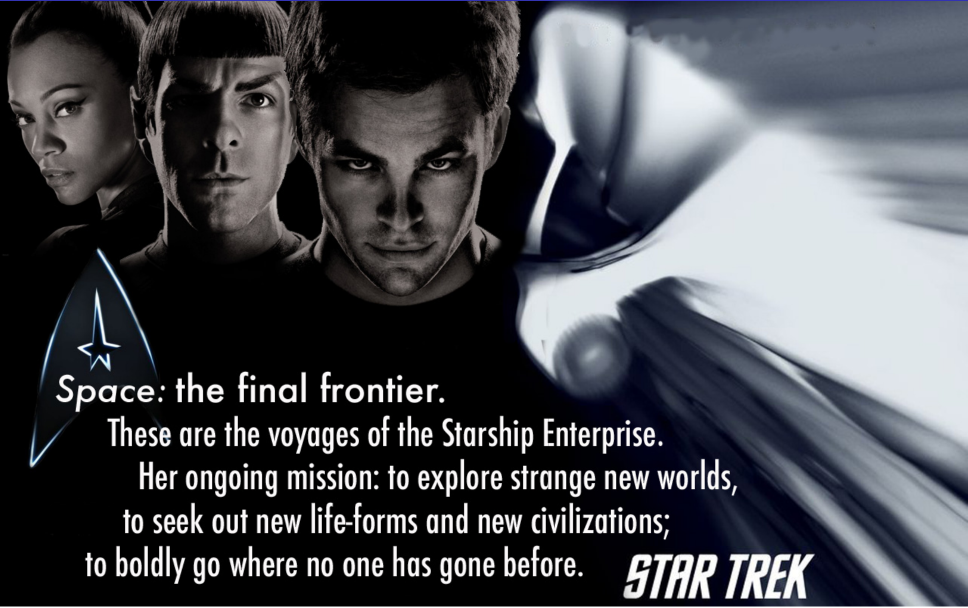
Instructor: David Earn

Lecture 19
Space
Wednesday 28 February 2018

Announcments

- **Assignment 3** due today.
  - Do group contribution survey TODAY!!
- **Assignment 4** due Monday 12 March 2018, 11:30am.
- **Midterm test:**
  - *Date:* Thursday 8 March 2018
  - *Time:* 7:00pm to 9:00pm
  - *Location:* BSB-B154

Spatial Epidemic Dynamics



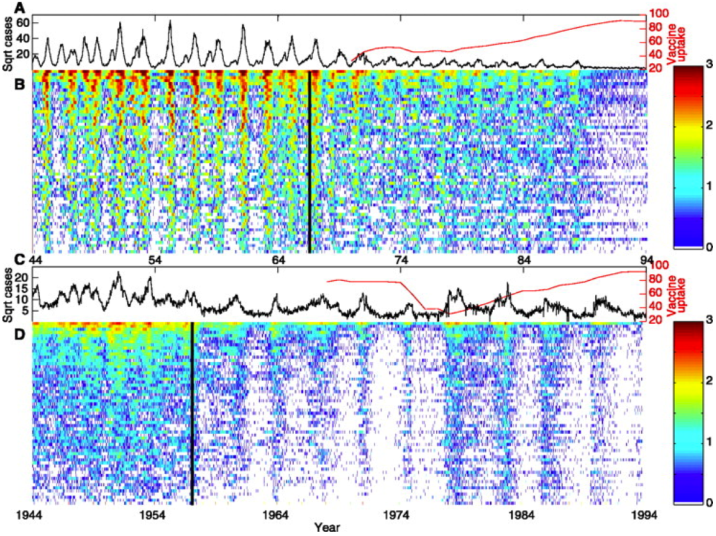
Something to think about

- All of our analysis has been of temporal patterns of epidemics
- What about spatial patterns?
- What problems are suggested by observed spatial epidemic patterns?
- Can spatial epidemic data suggest improved strategies for control?
- Can we reduce the eradication threshold below $p_{\text{crit}} = 1 - \frac{1}{\mathcal{R}_0}$?

Measles and Whooping Cough in 60 UK cities

Measles

Whooping Cough



Rohani, Earn & Grenfell (1999) *Science* **286**, 968–971

Better Control? Eradication?

- The term-time forced SEIR model successfully predicts past patterns of epidemics of childhood diseases
- Can we manipulate epidemics predictably so as to increase probability of eradication?
- Can we eradicate measles?

Idea for eradicating measles

- Try to re-synchronize measles epidemics in the UK and, moreover, synchronize measles epidemics worldwide: synchrony is good
- Devise new vaccination strategy that tends to synchronize...
- Avoid spatially structured epidemics...
- Time to think about the mathematics of synchrony...
- But analytical theory of synchrony in a periodically forced system of differential equations is mathematically demanding...
- So let's consider a much simpler biological model...

The Logistic Map

Logistic Map

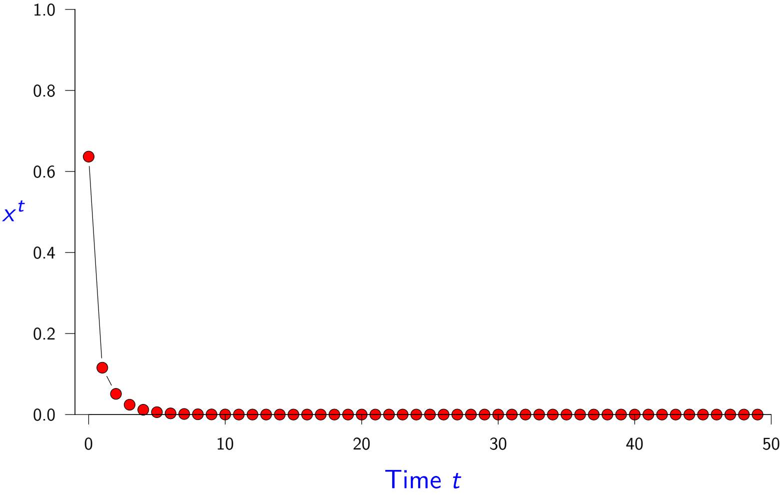
- Simplest non-trivial *discrete time* population model for a single species (with *non-overlapping generations*) in a *single habitat patch*.
- Time: $t = 0, 1, 2, 3, \ldots$
- State: $x \in [0, 1]$ (population density)
- Population density at time t is x^t. Solutions are sequences:

$$x^0,\ x^1,\ x^2,\ \ldots$$

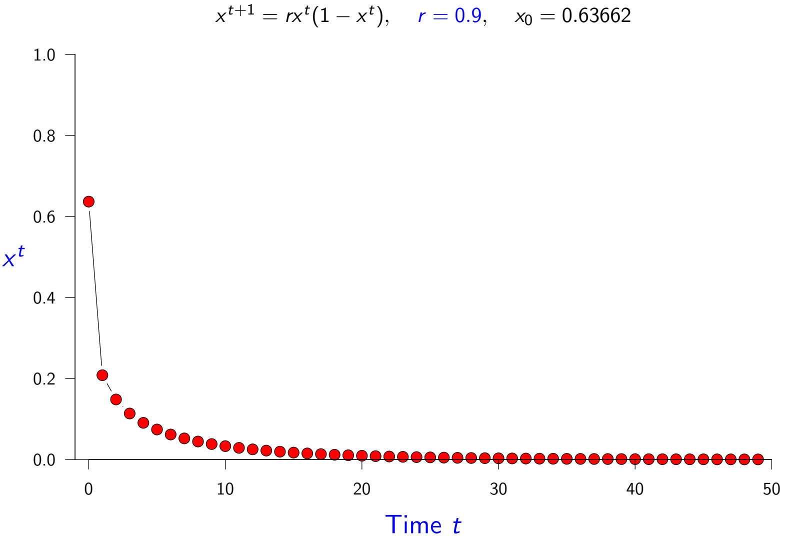
- $x^{t+1} = F(x^t)$ for some *reproduction function* $F(x)$.
- For logistic map: $F(x) = rx(1-x)$, so $x^{t+1} = rx^t(1-x^t)$. $x^{t+1} = [r(1-x^t)]x^t \implies r$ is *maximum fecundity* (which is achieved in limit of very small population density).
- What kinds of dynamics are possible for the Logistic Map?

Logistic Map Time Series, $r = 0.5$

$$x^{t+1} = rx^t(1 - x^t), \quad r = 0.5, \quad x_0 = 0.63662$$

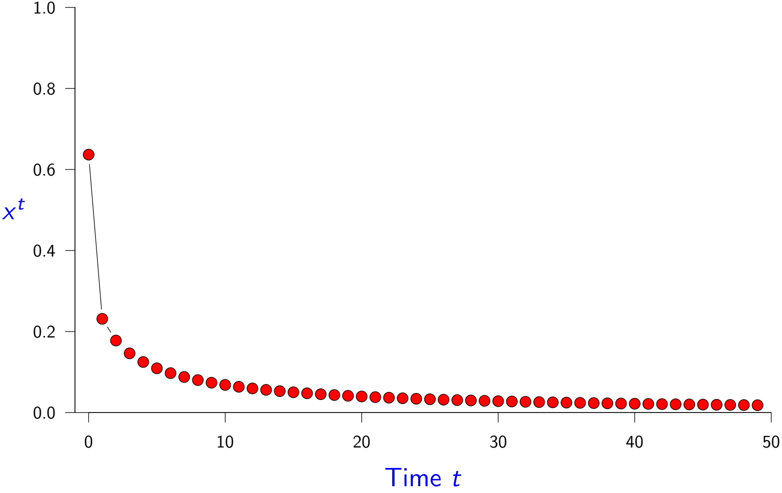


Logistic Map Time Series, $r = 0.9$

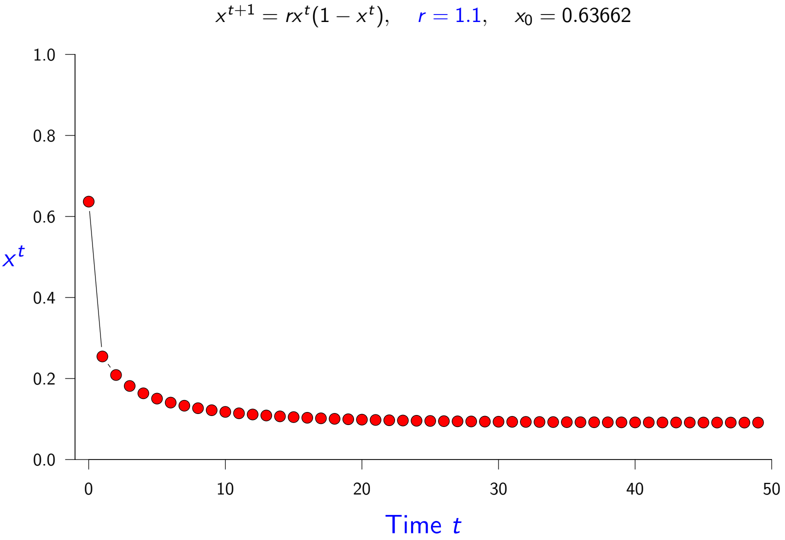


Logistic Map Time Series, $r = 1$

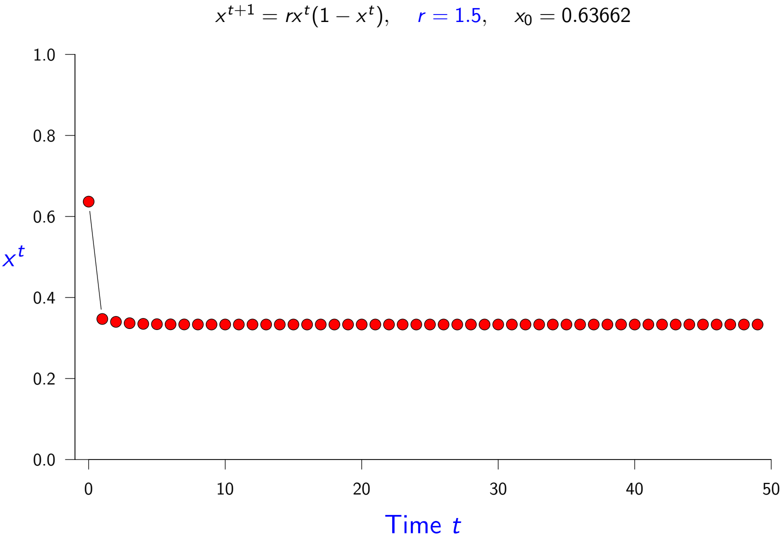
$$x^{t+1} = rx^t(1 - x^t), \quad r = 1, \quad x_0 = 0.63662$$



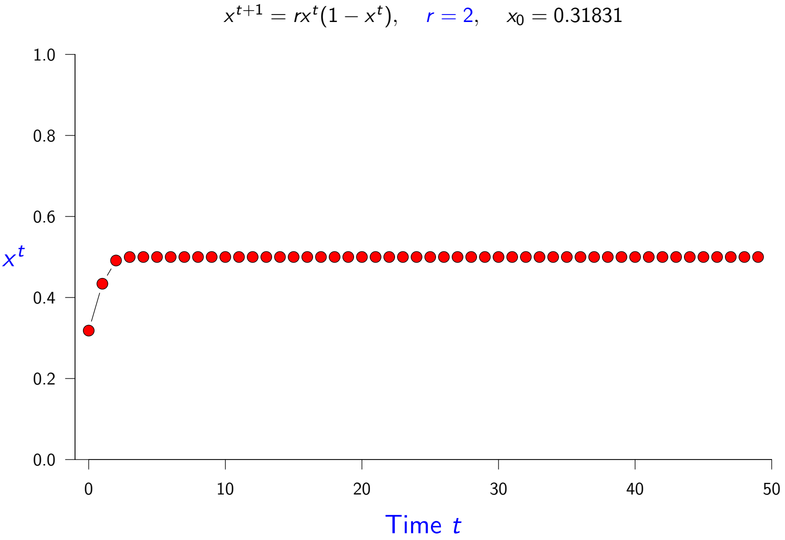
Logistic Map Time Series, $r = 1.1$



Logistic Map Time Series, $r = 1.5$

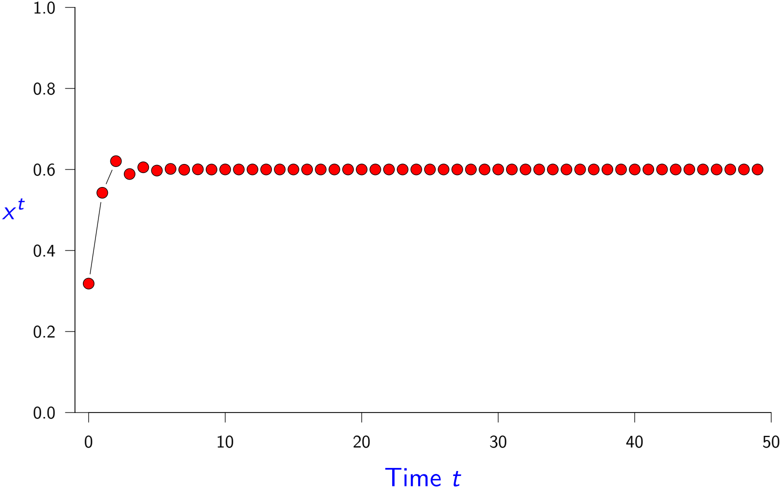


Logistic Map Time Series, $r = 2$



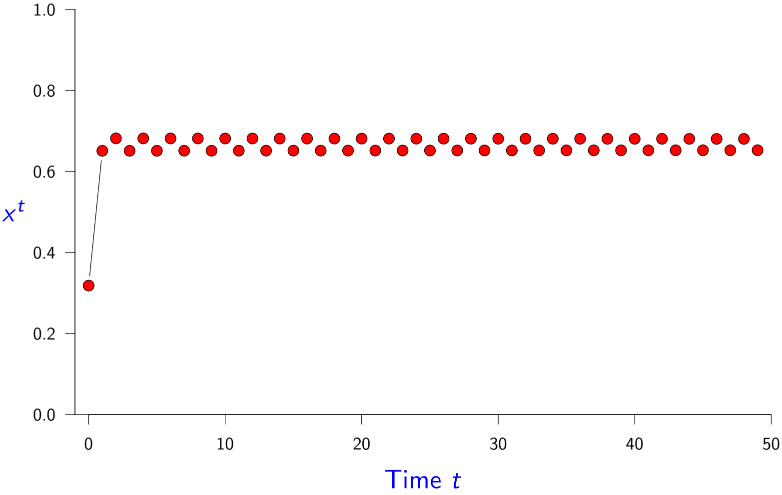
Logistic Map Time Series, $r = 2.5$

$$x^{t+1} = rx^t(1 - x^t), \quad r = 2.5, \quad x_0 = 0.31831$$



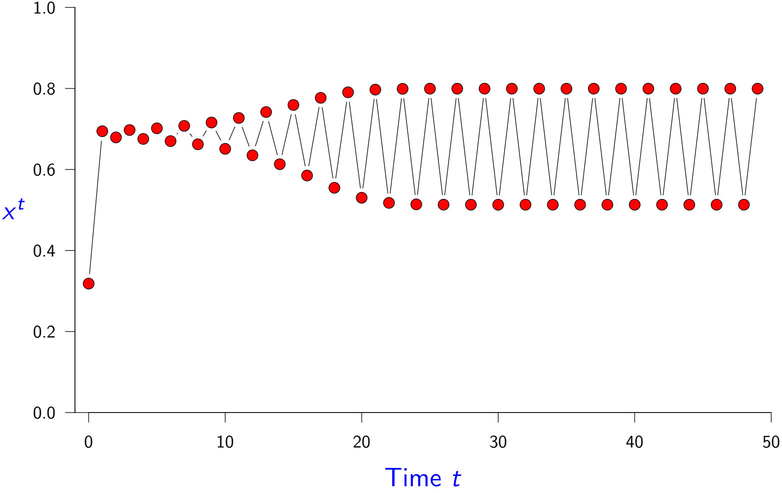
Logistic Map Time Series, $r = 3$

$$x^{t+1} = rx^t(1 - x^t), \quad r = 3, \quad x_0 = 0.31831$$



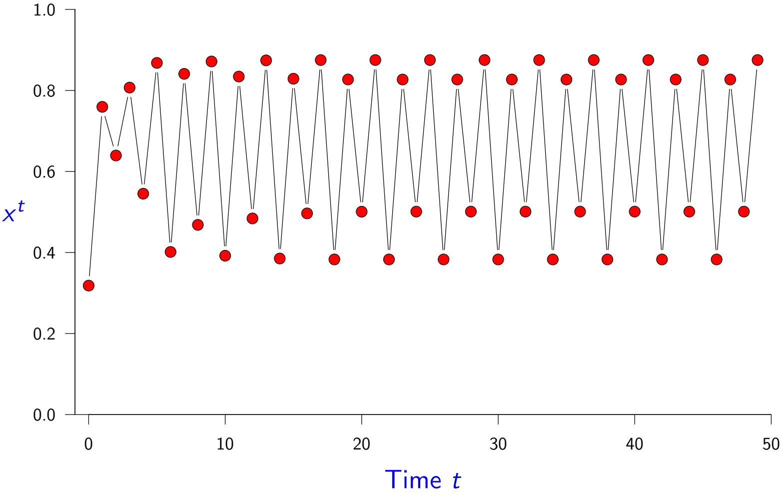
Logistic Map Time Series, $r = 3.2$

$$x^{t+1} = rx^t(1 - x^t), \quad r = 3.2, \quad x_0 = 0.31831$$



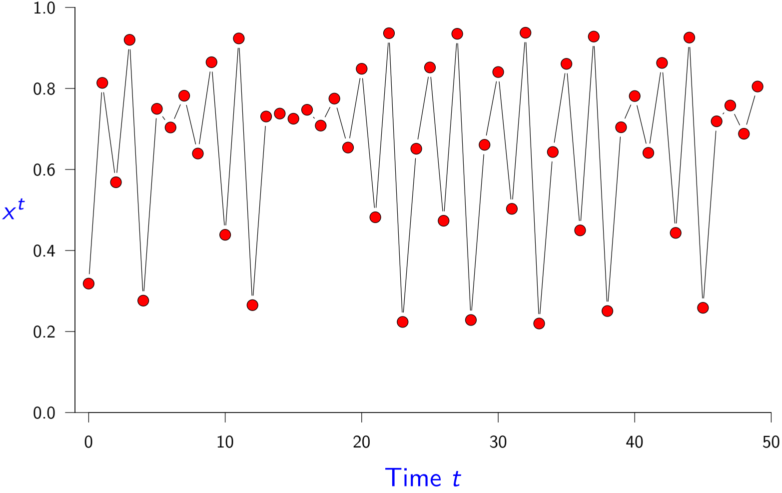
Logistic Map Time Series, $r = 3.5$

$$x^{t+1} = rx^t(1 - x^t), \quad r = 3.5, \quad x_0 = 0.31831$$



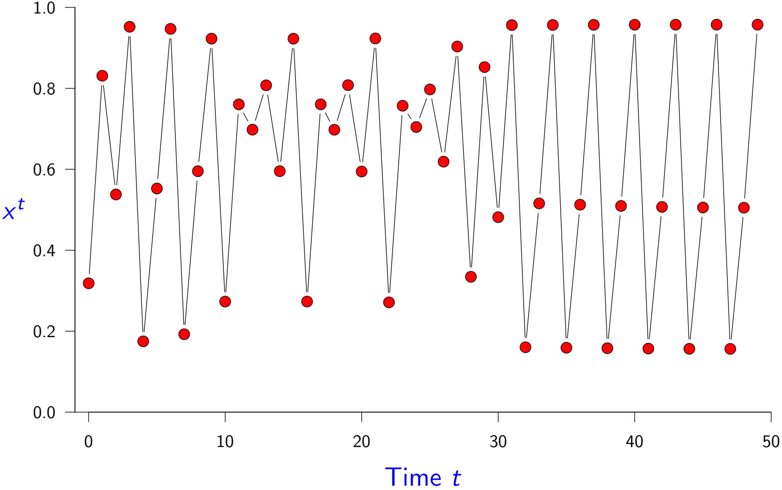
Logistic Map Time Series, $r = 3.75$

$x^{t+1} = rx^t(1 - x^t), \quad r = 3.75, \quad x_0 = 0.31831$

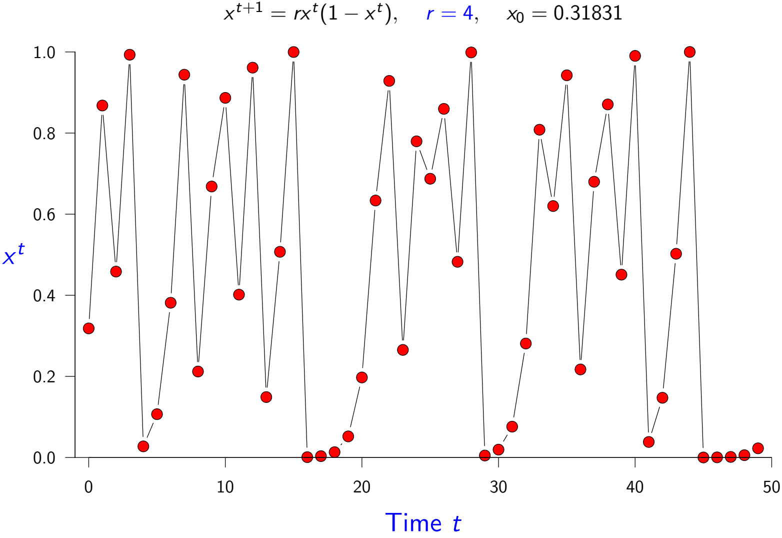


Logistic Map Time Series, $r = 3.83$

$$x^{t+1} = rx^t(1-x^t), \quad r = 3.83, \quad x_0 = 0.31831$$



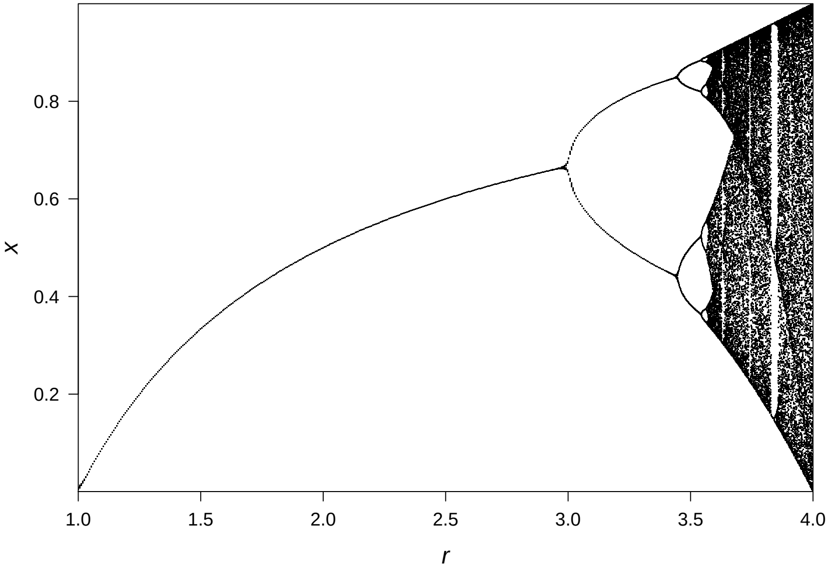
Logistic Map Time Series, $r = 4$



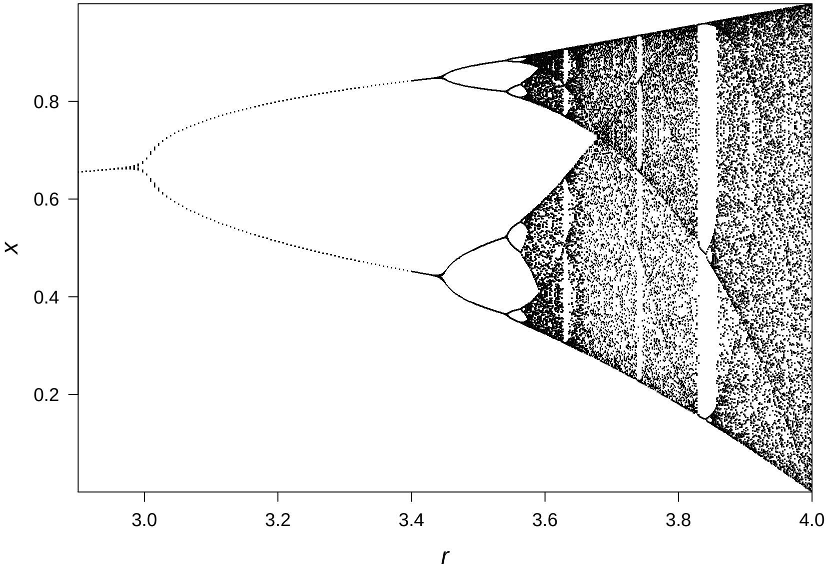
Logistic Map Summary

- Time series show:
  - $r \leq 1 \implies$ Extinction.
  - $1 < r < 3 \implies$ Persistence at equilibrium.
  - $r > 3 \implies$ period doubling cascade to chaos, then appearance of cycles of all possible lengths, and more chaos, ...
- How can we summarize this in a diagram?
  - Bifurcation diagram (wrt r).
  - Ignore transient behaviour: just show attractor.

Logistic Map, $F(x) = rx(1-x), \quad 1 \le r \le 4$



Logistic Map, $F(x) = rx(1-x)$, $2.9 \le r \le 4$



Logistic Map, $F(x) = rx(1-x), \quad 3.4 \le r \le 4$