McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 4MB3/6MB3 Mathematical Biology

Instructor: David Earn

Lecture 8
Epidemic Data
Wednesday 24 January 2018

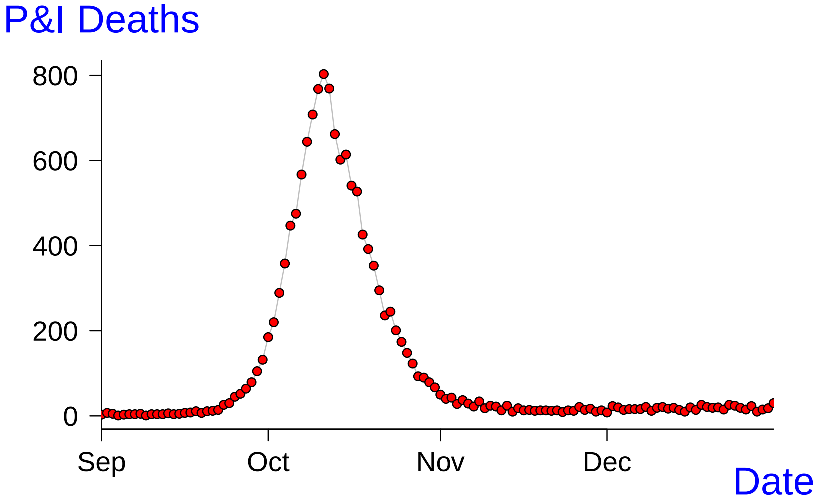
Announcements

- Thanks everyone for doing the contributions survey for Assignment 1.
- Don't stress about the ratings about each other's contributions. The issue is whether some group members did not pull their weight. If somebody didn't try and others had to pick up the slack, that person should be penalized. I will not penalize somebody because they tried but felt they didn't contribute as much to the final document as they could have. Do try to even out the work across the assignments.
- Make sure everyone in your group gets a chance to be in control of the LaTeX for one assignment.

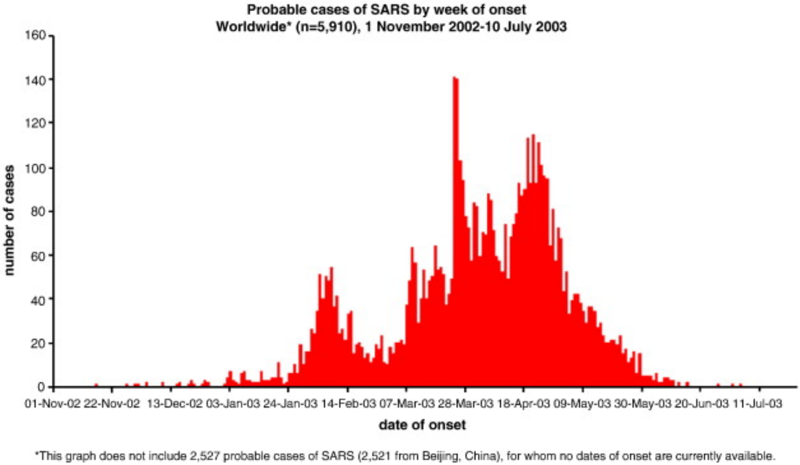
More Announcements!

- **Assignment 2:**
  Due Monday 5 February 2018 in class (and by e-mail) at 11:30am.
- **Midterm test:**
  - *Date:* week of 5–9 March? or 12–16 March?
  - *Time:* TBA
  - *Location:* TBA

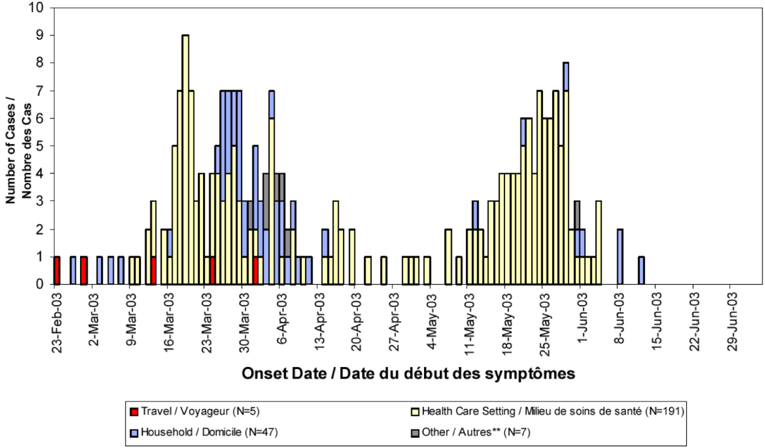
P&I Mortality, Philadelphia, 1918



SARS in 2003 (Worldwide)



SARS in 2003 (Toronto)

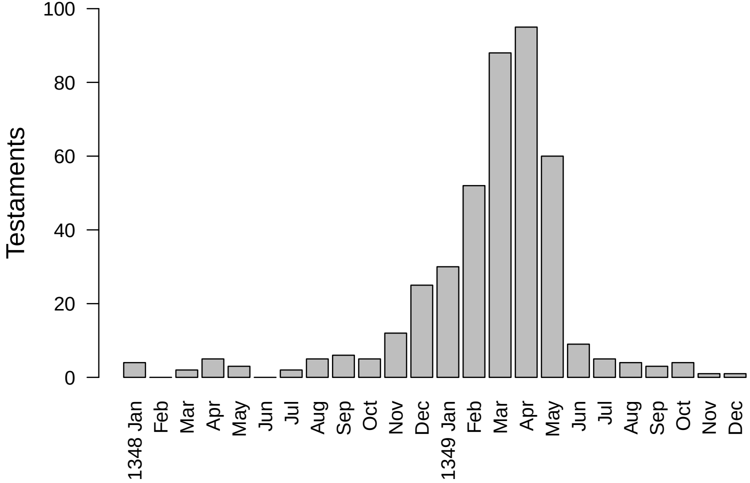


$N = 249$ (of 250 reported)

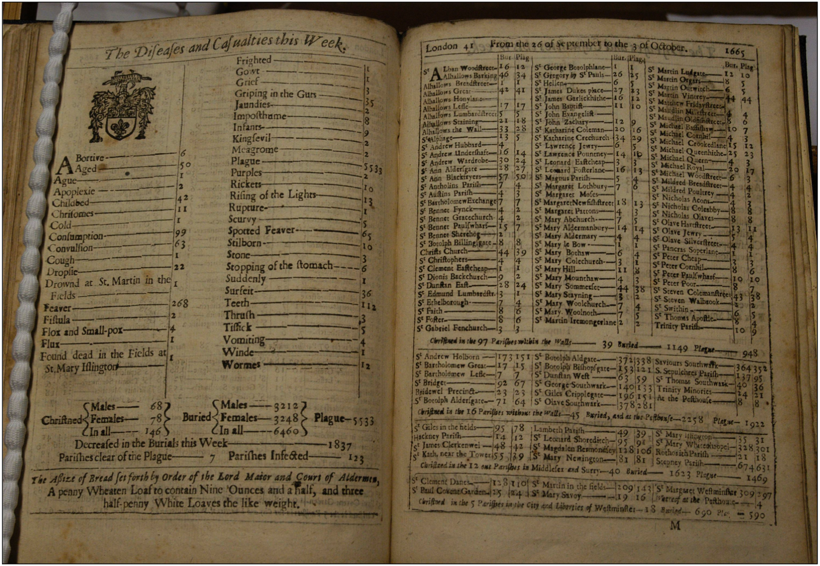
Some SARS Facts

- High case fatality
  - 1918 flu $< 3\%$
  - SARS $> 10\%$
- Long hospital stays
  - Mean time from admission to discharge or death: $\sim$ 25 days in Hong Kong
- 8098 probable cases, 774 deaths
- How bad would it have been if it had not been controlled?

The Black Death in London, England, 1348–1349



London Bill of Mortality, 26 Sept to 3 Oct 1665



The Diseases and Casualties this Week.

Abortive	6	Frighted	1
Aged	50	Gowt	1
Ague	1	Grief	3
Apoplexie	2	Griping in the Guts	35
Childbed	42	Jaundies	2
Chrisomes	11	Impostume	8
Cold	1	Infants	9
Consumption	99	Kingsevil	2
Convulsion	63	Meagrome	2
Cough	1	Plague	5533
Dropsie	22	Purples	2
Drowned at St. Martin in the Fields	1	Rickets	10
Feaver	268	Rising of the Lights	13
Fistula	2	Rupture	1
Flox and Small-pox	4	Scurvy	5
Flux	1	Spotted Feaver	65
Found dead in the Fields at St. Mary Islington	1	Stilborn	10
		Stone	3
		Stopping of the stomach	6
		Suddenly	1
		Surfeit	36
		Teeth	112
		Thrush	3
		Tissick	5
		Vomiting	4
		Winde	1
		Wormes	12

Christned: Males 68, Females 78, In all 146. Buried: Males 3212, Females 3248, In all 6460. Plague 5533.

Decreased in the Burials this Week 1837

Parishes clear of the Plague 7 Parishes Infected 123

The Assize of Bread set forth by Order of the Lord Maior and Court of Aldermen, A penny Wheaten Loaf to contain Nine Ounces and a half, and three half-penny White Loaves the like weight.

London 41 From the 26 of September to the 3 of October. 1665

	Bur.	Plag.		Bur.	Plag.		Bur.	Plag.
St Alban Woodstreet	16	12	St George Botolphlane	1	1	St Martin Ludgate	12	10
Alhallows Barking	46	34	St Gregory by St Pauls	26	25	St Martin Orgars	8	5
Alhallows Breadstreet	1	1	St Hellen	6	5	St Martin Outwitch	6	5
Alhallows Great	42	41	St James Dukes place	27	23	St Martin Vintrey	44	44
Alhallows Honylane			St James Garlickhithe	16	12	St Matthew Fridaystreet		
Alhallows Lesse	17	17	St John Baptist	11	10	St Maudlin Milkstreet	4	4
Alhallows Lumbardstreet	5	5	St John Evangelist			St Maudlin Oldfishstreet	8	6
Alhallows Staining	21	18	St John Zachary	12	9	St Michael Bassishaw	10	7
Alhallows the Wall	33	28	St Katharine Coleman	20	16	St Michael Cornhil	4	3
St Alphage	13	5	St Katharine Crechurch	34	29	St Michael Crookedlane	15	12
St Andrew Hubbard	4		St Lawrence Jewry	6	5	St Michael Queenhithe	25	23
St Andrew Undershaft	16	14	St Lawrence Pountney	14	10	St Michael Quern	4	3
St Andrew Wardrobe	30	24	St Leonard Eastcheap	3	3	St Michael Royal	20	17
St Ann Aldersgate	28	27	St Leonard Fosterlane	16	13	St Michael Woodstreet	6	3
St Ann Blackfryers	57	50	St Magnus Parish	5	4	St Mildred Breadstreet	4	4
St Antholins Parish	7	4	St Margaret Lothbury	7	6	St Mildred Poultrey	4	2
St Austins Parish	4	3	St Margaret Moses			St Nicholas Acons	4	3
St Bartholomew Exchange	7	7	St Margaret Newfishstreet	18	13	St Nicholas Coleabby	8	8
St Bennet Fynck	4	2	St Margaret Pattons	4	3	St Nicholas Olaves	8	8
St Bennet Gracechurch	4	2	St Mary Abchurch	7	5	St Olave Hartstreet	13	11
St Bennet Paulswharf	15	7	St Mary Aldermanbury	14	14	St Olave Jewry	5	4
St Bennet Sherehog	2		St Mary Aldermary	4	4	St Olave Silverstreet	4	4
St Botolph Billingsgate	8	8	St Mary le Bow	1	1	St Pancras Soperlane	1	1
Christs Church	44	39	St Mary Bothaw	6	4	St Peter Cheap	3	3
St Christophers	4	4	St Mary Colechurch	3	1	St Peter Cornhil	8	6
St Clement Eastcheap	1	1	St Mary Hill	11	8	St Peter Paulswharf	10	10
St Dionis Backchurch	9	2	St Mary Mounthaw	4	3	St Peter Poor	8	7
St Dunstan East	28	24	St Mary Sommerset	44	38	St Steven Colemanstreet	43	38
St Edmund Lumbardstr.	3	1	St Mary Stayning	3	2	St Steven Walbrook	2	2
St Ethelborough	7	4	St Mary Woolchurch	7	4	St Swithin	6	5
St Faith	8	6	St Mary Woolnoth	7	5	St Thomas Apostle	8	4
St Foster	8	6	St Martin Iremongerlane	2	2	Trinity Parish	10	9
St Gabriel Fenchurch	3	3						

Christned in the 97 Parishes within the Walls — 39 *Buried* — 1149 *Plague* — 948

St Andrew Holborn	173	151	St Botolph Aldgate	372	338	Saviours Southwark	364	352
St Bartholomew Great	17	15	St Botolph Bishopsgate	153	121	S. Sepulchres Parish	137	95
St Bartholomew Lesse	7	7	St Dunstan West	63	59	St Thomas Southwark	40	36
St Bridget	92	67	St George Southwark	140	133	Trinity Minories	24	21
Bridewel Precinct	23	23	St Giles Cripplegate	196	151	At the Pesthouse	8	8
St Botolph Aldersgate	71	64	St Olave Southwark	378	281			

Christned in the 16 Parishes without the Walls — 45 *Buried, and at the Pesthouse* — 2258 *Plague* — 1922

St Giles in the fields	95	78	Lambeth Parish	49	39	St Mary Islington	35	31
Hackney Parish	14	12	St Leonard Shoreditch	95	91	St Mary Whitechappel	328	301
St James Clerkenwel	48	42	St Magdalen Bermondsey	128	106	Rothorith Parish	21	18
St Kath. near the Tower	55	39	St Mary Newington	81	81	Stepney Parish	674	631

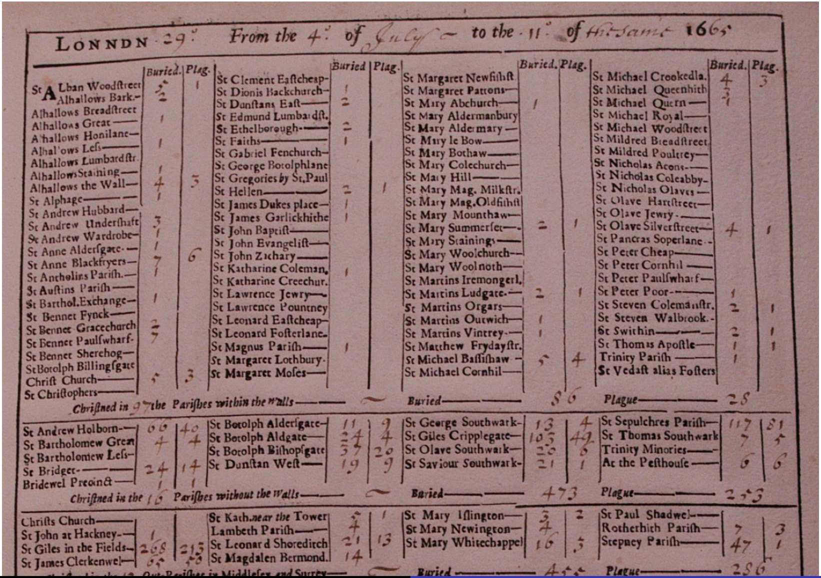
Christned in the 12 out Parishes in Middlesex and Surry — 40 *Buried* — 1623 *Plague* — 1469

St Clement Danes	128	110	St Martin in the fields	209	143	St Margaret Westminster	309	297
St Paul Covent Garden	25	24	St Mary Savoy	19	16	Whereof at the Pesthouse		4

Christned in the 5 Parishes in the City and Liberties of Westminster — 18 *Buried* — 690 *Plague* — 590

M

Mortality Bills are typically handwritten



LONNDN 29° From the 4° of July to the 11° of the same 1665

Parish	Buried	Plag.	Parish	Buried	Plag.	Parish	Buried	Plag.	Parish	Buried	Plag.
St Alban Woodſtreet	2	1	St Clement Eaſtcheap	1		St Margaret Newfiſhſt			St Michael Crookedla.	4	3
Alhallows Bark	2		St Dionis Backchurch	2		St Margaret Pattons			St Michael Queenhith	3	
Alhallows Breadſtreet			St Dunſtans Eaſt			St Mary Abchurch	1		St Michael Quern	1	
Alhallows Great	1		St Edmund Lumbardſt.			St Mary Aldermanbury			St Michael Royal		
Alhallows Honilane			St Ethelborough	2		St Mary Aldermary			St Michael Woodſtreet		
Alhallows Leſs	1		St Faiths	1		St Mary le Bow			St Mildred Breadſtreet		
Alhallows Lumbardſtr.			St Gabriel Fenchurch			St Mary Bothaw			St Mildred Poultrey		
Alhallows Staining	1		St George Botolphlane			St Mary Colechurch			St Nicholas Acons		
Alhallows the Wall	4	3	St Gregories by St. Paul			St Mary Hill			St Nicholas Coleabby		
St Alphage	1		St Hellen	2	1	St Mary Mag. Milkſtr.			St Nicholas Olaves		
St Andrew Hubbard			St James Dukes place	1		St Mary Mag. Oldfiſhſt			St Olave Hartſtreet		
St Andrew Underſhaft	3		St James Garlickhithe	1		St Mary Mounthaw			St Olave Jewry		
St Andrew Wardrobe	1		St John Baptiſt			St Mary Summerſet	2	1	St Olave Silverſtreet	4	1
St Anne Alderſgate	1		St John Evangeliſt			St Mary Stainings			St Pancras Soperlane		
St Anne Blackfryers	7	6	St John Zachary			St Mary Woolchurch			St Peter Cheap		
St Antholins Pariſh	1		St Katharine Coleman	1		St Mary Woolnoth			St Peter Cornhil		
St Auſtins Pariſh			St Katharine Creechur.			St Martins Iremongerl.			St Peter Paulſwharf		
St Barthol. Exchange	1		St Lawrence Jewry			St Martins Ludgate	2	1	St Peter Poor	1	
St Bennet Fynck			St Lawrence Pountney			St Martins Orgars			St Steven Colemanſtr.	2	1
St Bennet Gracechurch	2		St Leonard Eaſtcheap			St Martins Outwich	1		St Steven Walbrook		
St Bennet Paulſwharf	7		St Leonard Foſterlane			St Martins Vintrey	1		St Swithin	2	1
St Bennet Sherehog			St Magnus Pariſh	1		St Matthew Frydayſtr.			St Thomas Apoſtle	1	1
St Botolph Billingſgate			St Margaret Lothbury			St Michael Baſſiſhaw	5	4	Trinity Pariſh	1	
Chriſt Church	5	3	St Margaret Moſes			St Michael Cornhil			St Vedaſt alias Foſters		
St Chriſtophers											

Chriſtned in 97 the Pariſhes within the walls — Buried — 86 Plague — 28

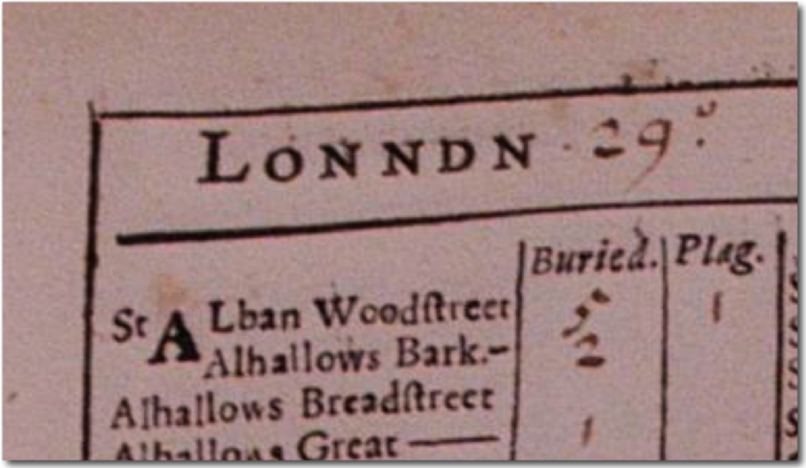
Parish	Buried	Plag.	Parish	Buried	Plag.	Parish	Buried	Plag.	Parish	Buried	Plag.
St Andrew Holborn	66	40	St Botolph Alderſgate	11	9	St George Southwark	13	4	St Sepulchres Pariſh	117	81
St Bartholomew Great	4	4	St Botolph Aldgate	24	4	St Giles Cripplegate	103	42	St Thomas Southwark	7	5
St Bartholomew Leſs			St Botolph Biſhopſgate	37	20	St Olave Southwark	20	6	Trinity Minories		
St Bridget	24	14	St Dunſtan Weſt	19	9	St Saviour Southwark	21	1	At the Peſthouſe	6	6
Bridewel Precinct	1	1									

Chriſtned in the 16 Pariſhes without the walls — Buried — 473 Plague — 253

Parish	Buried	Plag.	Parish	Buried	Plag.	Parish	Buried	Plag.	Parish	Buried	Plag.
Chriſts Church			St Kath. near the Tower	5	1	St Mary Iſlington	3	2	St Paul Shadwel		
St John at Hackney	1		Lambeth Pariſh	4		St Mary Newington	4		Rotherhith Pariſh	7	3
St Giles in the Fields	268	213	St Leonard Shoreditch	21	13	St Mary Whitechappel	16	5	Stepney Pariſh	47	1
St James Clerkenwel	65	58	St Magdalen Bermond.	14							

… in the 12 Out-Pariſhes in Middleſex and Surrey — Buried 455 Plague 256

But handwriting is usually very clear



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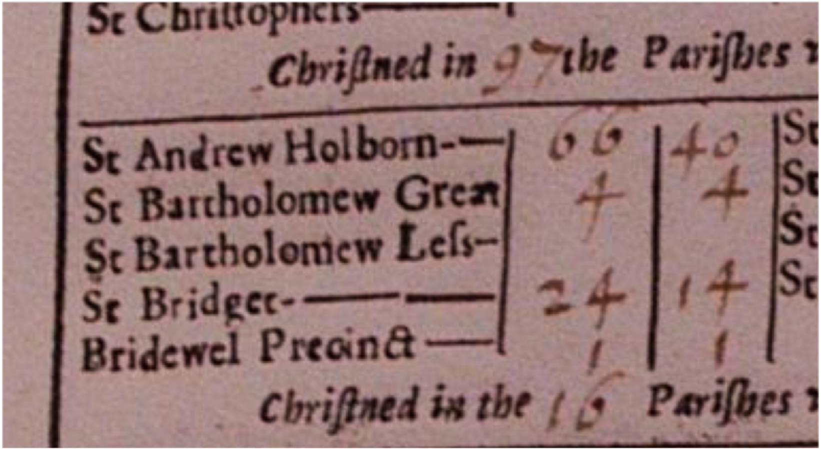
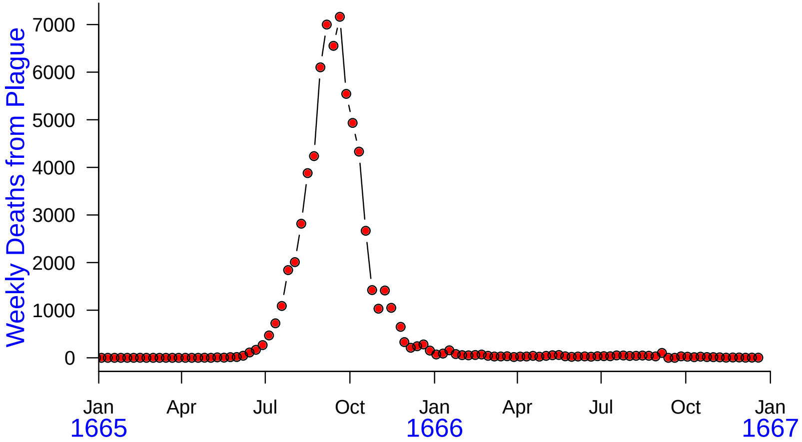
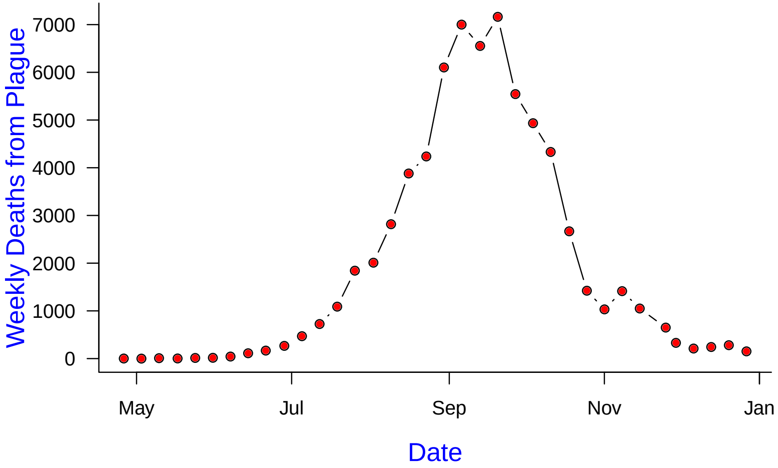

St Christophers
Chrisned in 97 the Parishes
St Andrew Holborn 66 40
St Bartholomew Great 4 4
St Bartholomew Less
St Bridget 24 14
Bridewel Precinct 1 1
Chrisned in the 16 Parishes

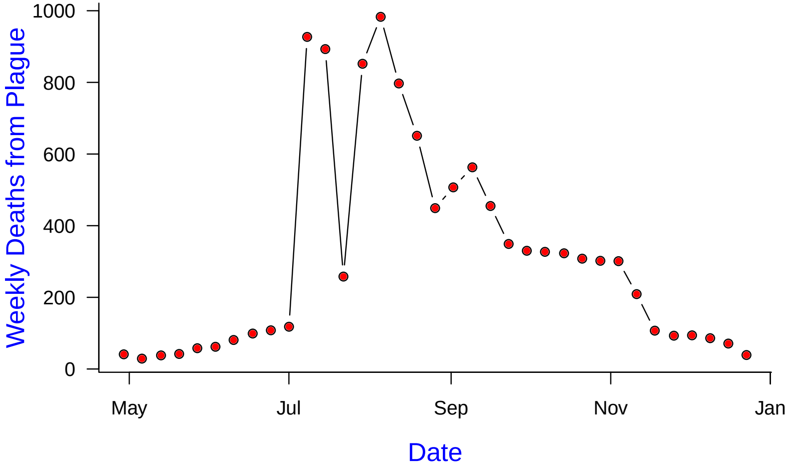
The Great Plague of London, 1665



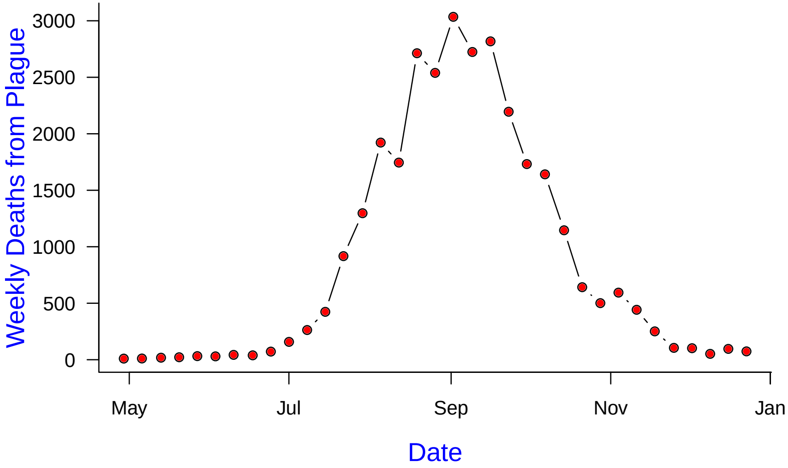
The Great Plague of London, 1665



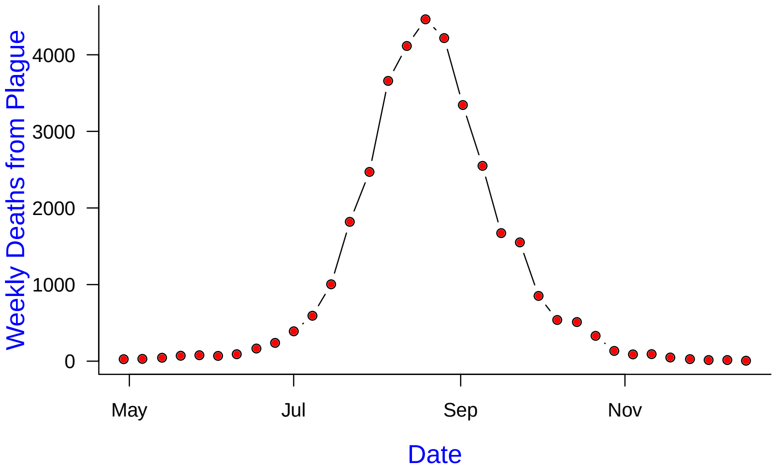
London Plague of 1593



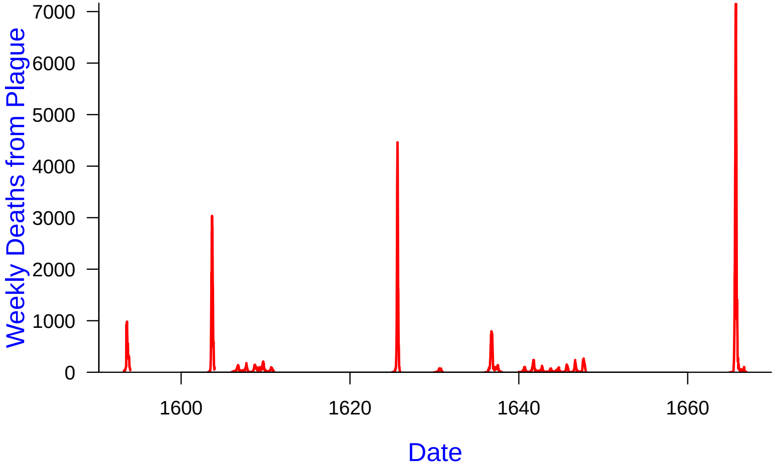
London Plague of 1603



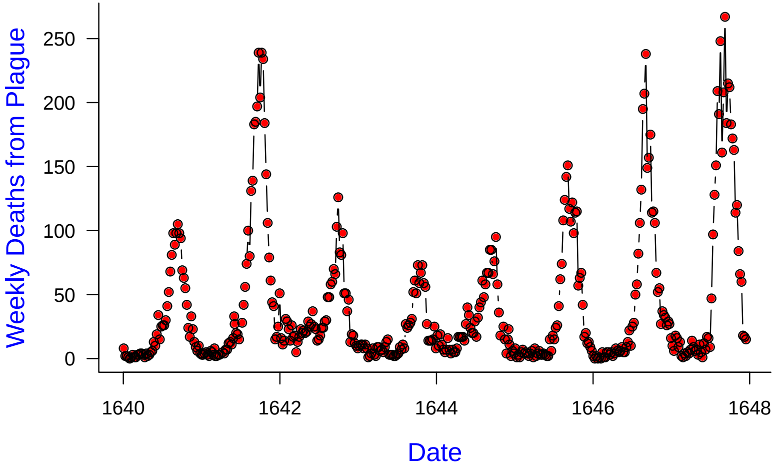
London Plague of 1625



Weekly Deaths from Plague in London, 1592–1666



Weekly Plague in London, 1640–1648



Some Plague Facts

- Plague epidemics recorded from Roman times to early 1900s.
- $\gtrsim 1/3$ Europe's population died in "Black Death" of 1348
  - ~ 300 years for the population to reach the same level.
- Recently (2011) established (at McMaster!) that the pathogen that caused The Black Death was *Yersinia pestis*

  [Bos *et al.* 2011, **Nature 478**, 506–510]

- More recently (2014) established (again at McMaster!) that the pathogen that caused The Plague of Justinian (541–543 AD) was *Yersinia pestis*

  [Wagner *et al.* 2014, **Lancet Infectious Diseases 14**, 319–326]

- *Y. pestis* still a concern?
  Yes: Rodent reservoir, antibiotic-resistant strains, bioterrorism
- **Spatial data** for any plagues? Yes, for London in 1665. . .

Visualization of spatial structure of Great Plague

- GIS encoding of parish boundaries
- Overlay parish boundaries on more modern map for reference
- Colour parishes as they become infected
- Is there evidence for spatial spread or was the spatial pattern random?
- DE low-tech animation...
- CBC high-tech animation...
  - *The Nature of Things*, 21 August 2014. http://www.cbc.ca/natureofthings/episodes/secrets-in-the-bones-the-hunt-for-the-black-death-killer

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 4MB3/6MB3 Mathematical Biology

Instructor: David Earn

Lecture 9
Epidemic Data II
Friday 26 Jan 2018

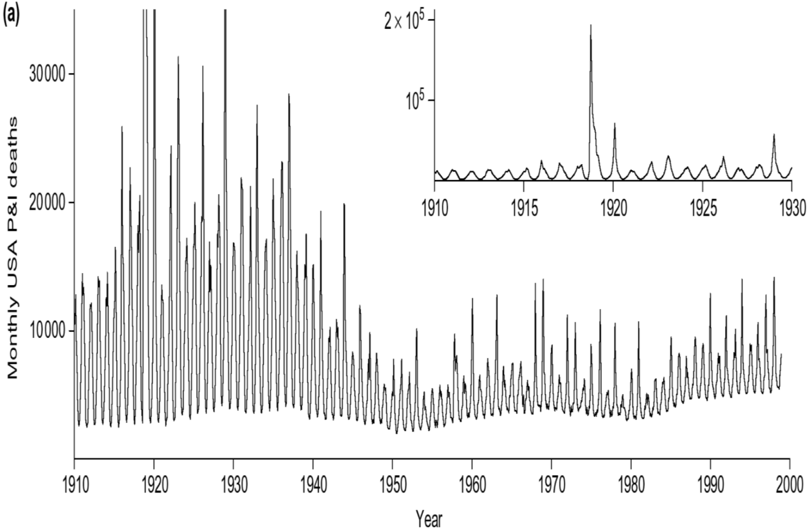
Announcements

- **Assignment 2:**
  Due Monday 5 February 2018 in class (and by e-mail) at 11:30am.
- **Midterm test:** We agreed on:
  - *Date:* Thursday 8 March 2018
  - *Time:* 7:00pm to 9:00pm
  - *Location:* TBA

Visualization of entire course of the Great Plague

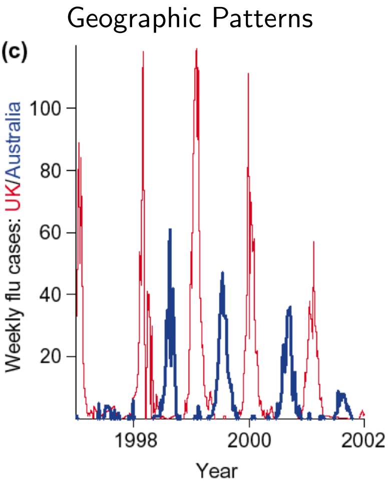
- What happenned after initial spatial spread?
- Visualize full spatial epidemic structure
- Show magnitude of epidemic in each parish with cylinder.
- Epidemic Visualization (`EpiVis`) software by Junling Ma.

P&I mortality in U.S.A., 1910–1998



Earn, Dushoff & Levin 2002, *Trends in Ecology and Evolution* **17**, 334–340

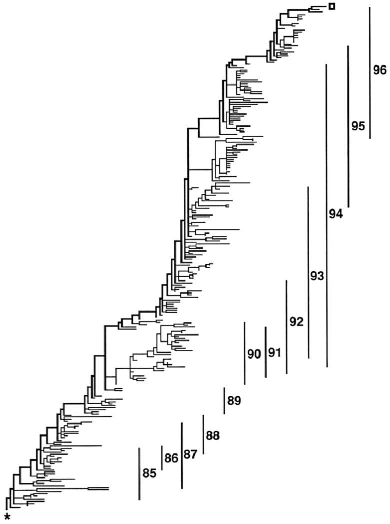
Influenza Incidence Patterns (lab confirmed)



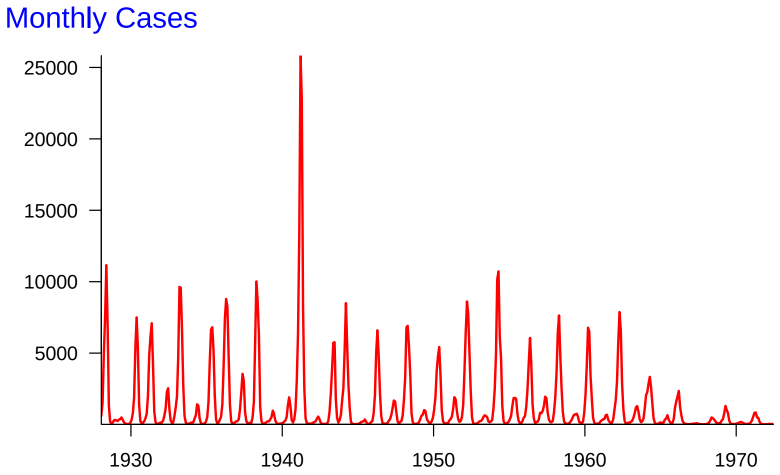
Earn, Dushoff & Levin 2002, *Trends in Ecology and Evolution* **17**, 334–340

Influenza Evolution

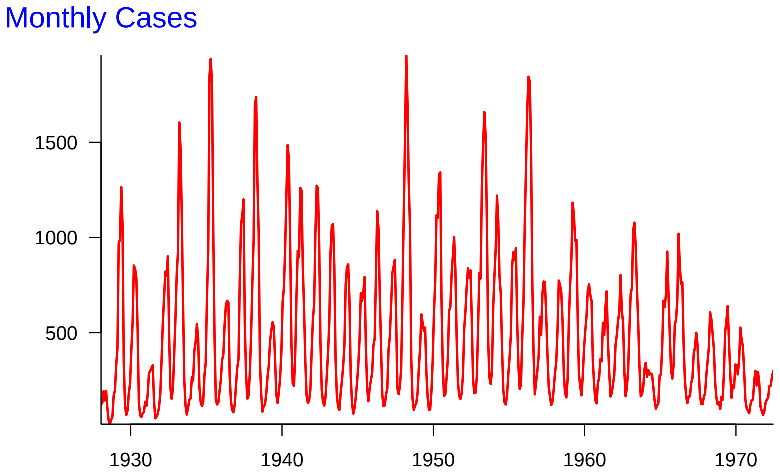
Molecular phylogenetic reconstruction of influenza A/H3N2 evolution, 1985–1996 (Fitch *et al.* 1997)



Measles in New York City, 1928–1972

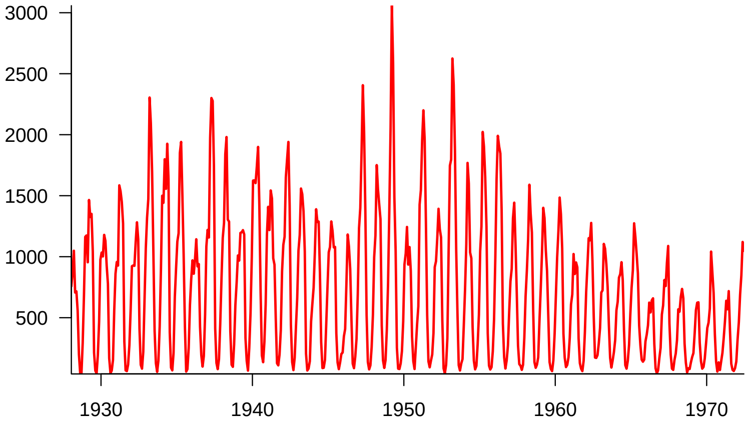


Mumps in New York City, 1928–1972

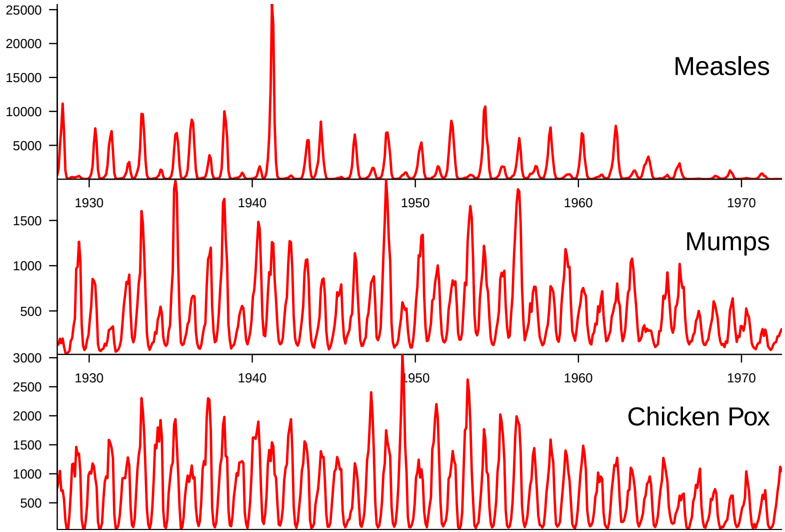


Chicken Pox in New York City, 1928–1972

Monthly Cases

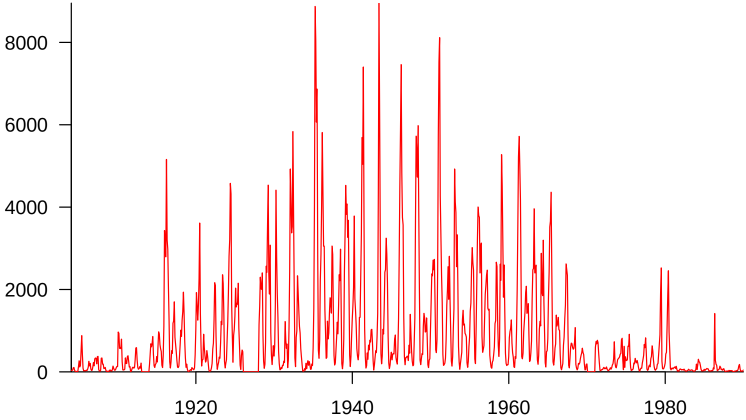


Childhood diseases in New York City, 1928–1972



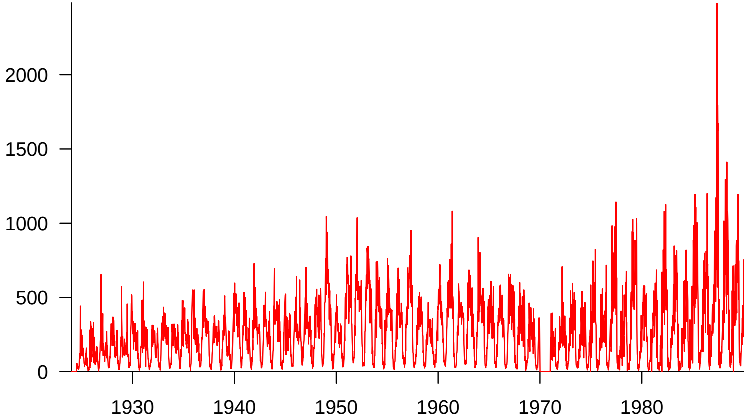
Measles in Ontario, 1904–1989

Monthly Cases



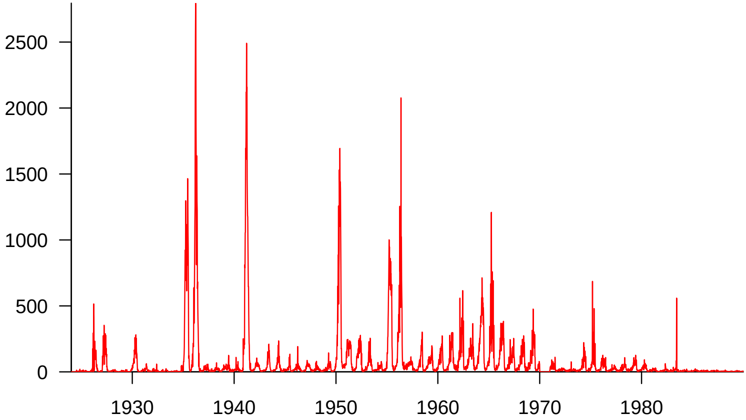
Chicken Pox in Ontario, 1924–1989

Monthly Cases



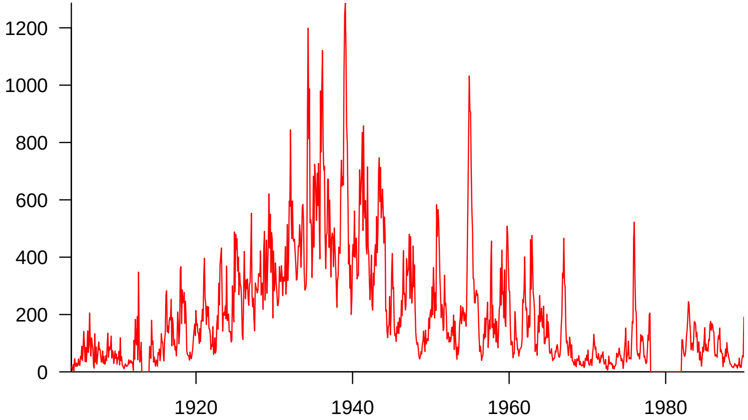
Rubella in Ontario, 1924–1989

Weekly Cases

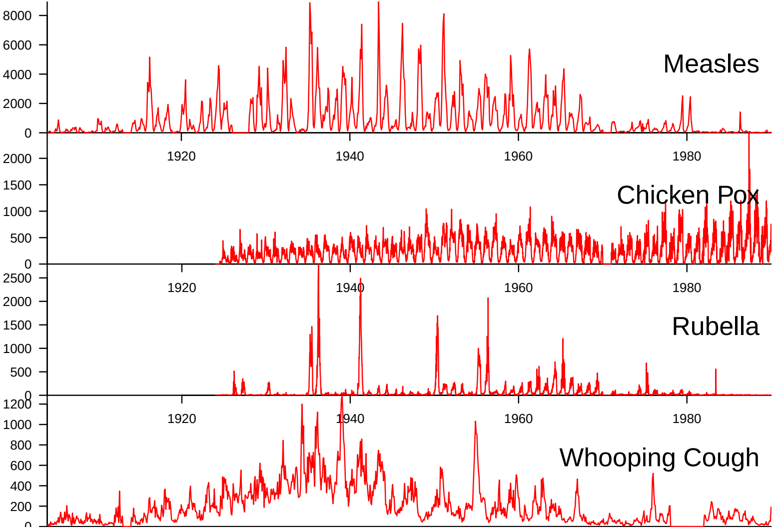


Whooping Cough in Ontario, 1904–1989

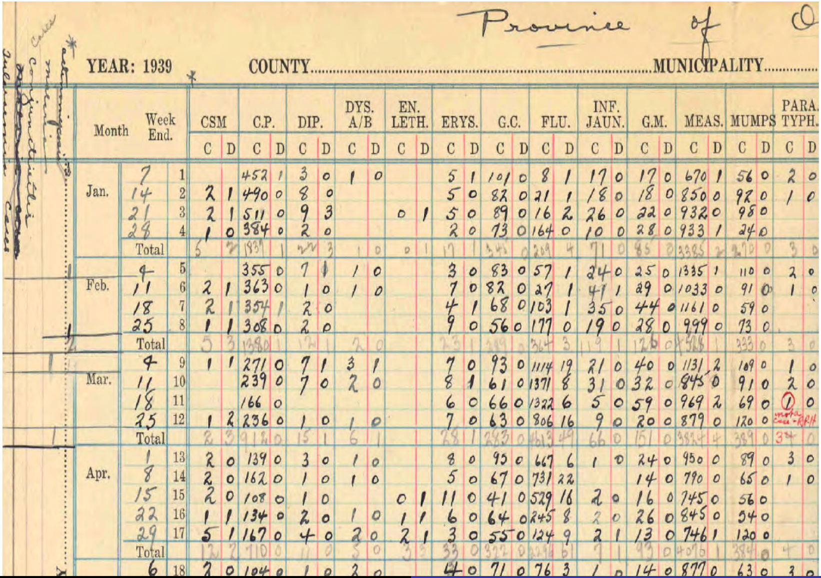
Monthly Cases



Childhood diseases in Ontario, 1904–1989



Ontario Disease Notification Data



Province of O…

YEAR: 1939 COUNTY.................... MUNICIPALITY............

Month	Week End.		CSM C	CSM D	C.P. C	C.P. D	DIP. C	DIP. D	DYS. A/B C	DYS. A/B D	EN. LETH. C	EN. LETH. D	ERYS. C	ERYS. D	G.C. C	G.C. D	FLU. C	FLU. D	INF. JAUN. C	INF. JAUN. D	G.M. C	G.M. D	MEAS. C	MEAS. D	MUMPS C	MUMPS D	PARA. TYPH. C	PARA. TYPH. D
Jan.	7	1			452	1	3	0	1	0			5	1	101	0	8	1	17	0	17	0	670	1	56	0	2	0
	14	2	2	1	490	0	8	0					5	0	82	0	21	1	18	0	18	0	850	0	92	0	1	0
	21	3	2	1	511	0	9	3			0	1	5	0	89	0	16	2	26	0	22	0	932	0	98	0		
	28	4	1	0	384	0	2	0					2	0	73	0	164	0	10	0	28	0	933	1	24	0		
	Total		5	2	1839	1	22	3	1	0	0	1	17	1	345	0	209	4	71	0	85	0	3385	2	270	0	3	0
Feb.	4	5			355	0	7	1	1	0			3	0	83	0	57	1	24	0	25	0	1335	1	110	0	2	0
	11	6	2	1	363	0	1	0	1	0			7	0	82	0	27	1	41	1	29	0	1033	0	91	0	1	0
	18	7	2	1	354	1	2	0					4	1	68	0	103	1	35	0	44	0	1161	0	59	0		
	25	8	1	1	308	0	2	0					9	0	56	0	177	0	19	0	28	0	999	0	73	0		
	Total		5	3	1380	1	12	1	2	0			23	1	289	0	364	3	119	1	126	0	4528	1	333	0	3	0
Mar.	4	9	1	1	271	0	7	1	3	1			7	0	93	0	1114	19	21	0	40	0	1131	2	109	0	1	0
	11	10			239	0	7	0	2	0			8	1	61	0	1371	8	31	0	32	0	845	0	91	0	2	0
	18	11			166	0							6	0	66	0	1322	6	5	0	59	0	969	2	69	0	1	0
	25	12	1	2	236	0	1	0	1	0			7	0	63	0	806	16	9	0	20	0	879	0	120	0		
	Total		2	3	912	0	15	1	6	1			28	1	283	0	4613	49	66	0	151	0	3824	4	389	0	3+	0
Apr.	1	13	2	0	139	0	3	0	1	0			8	0	95	0	667	6	1	0	24	0	950	0	89	0	3	0
	8	14	2	0	162	0	1	0	1	0			5	0	67	0	731	22			14	0	790	0	65	0	1	0
	15	15	2	0	108	0	1	0			0	1	11	0	41	0	529	16	2	0	16	0	745	0	56	0		
	22	16	1	1	134	0	2	0	1	0	1	1	6	0	64	0	245	8	2	0	26	0	845	0	54	0		
	29	17	5	1	167	0	4	0	2	0	2	1	3	0	55	0	124	9	2	1	13	0	746	1	120	0		
	Total		12	2	710	0	11	0	5	0	3	3	33	0	322	0	2296	61	7	1	93	0	4076	1	384	0	4	0
	6	18	2	0	104	0	1	0	2	0			4	0	71	0	76	3	1	0	14	0	877	0	63	0	3	0

Dominion Bureau of Statistics Disease Notification Data

VITAL STATISTICS BRANCH - COMMUNICABLE DISEASE SECTION

Cases of Whooping Cough Reported by Provincial Health Departments, Year 1924

WEEK ENDING	P.E.I.		N.S.		N.B.		QUE.		ONT.		MAN.		SASK.		ALTA.		B.C.		CANADA	
	WK	MT	WK	MT	WK	MT	WK	MT	WK	MT	WK	MT	WK	MT	WK	MT	WK	MT	WK	MT
1 JAN 5			11										1						12	
2 12			29										18						47	
3 19			37										32						69	
4 26			75	152				68		181		36	13	64		97		4	88	602
5 FEB 2			12		1								53						66	
6 9			5										40						45	
7 16			31										14						45	
8 23			-2	50	1	2		267		202		48	4	111		116		1	7	797
9 MAR 1			2										21						23	
10 8													9						9	
11 15			3										11						14	
12 22			60										34						94	
13 29			2	61				144		140		52	15	90		15		7	17	515
14 APR 5			9										11						20	
15 12			1										12						13	
16 19			26		1								8						35	
17 26			14	50	3	4		42		140		39	16	47		67		5	33	394
18 MAY 3			26										2						28	

McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 4MB3/6MB3
Mathematical Biology

Instructor: David Earn

Lecture 10
Epidemic Data III
Monday 29 Jan 2018

Announcements

- Comment from TA on Assignment 1:
  "For a few of the groups, I would recommend that they look over the work that their group members have done. Question 2c and 2d in particular were closely related and there were a few obvious cases where the students had not communicated with each other."

- **Assignment 2:**
  Due Monday 5 February 2018 in class (and by e-mail) at 11:30am.

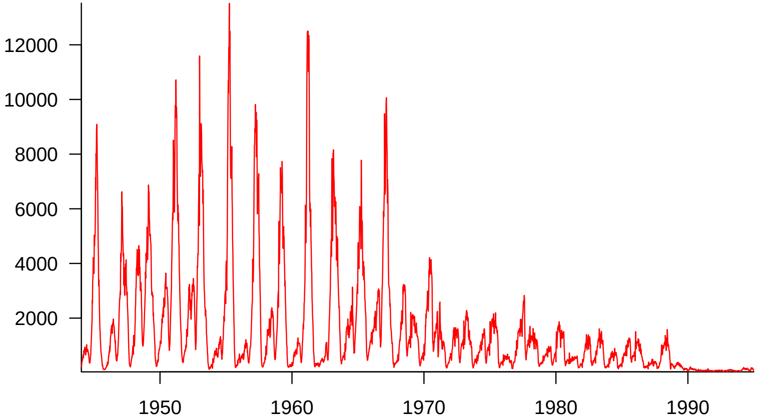
- **Midterm test:** We agreed on:
  - *Date:* Thursday 8 March 2018
  - *Time:* 7:00pm to 9:00pm
  - *Location:* TBA

Recurrent epidemics of childhood infections

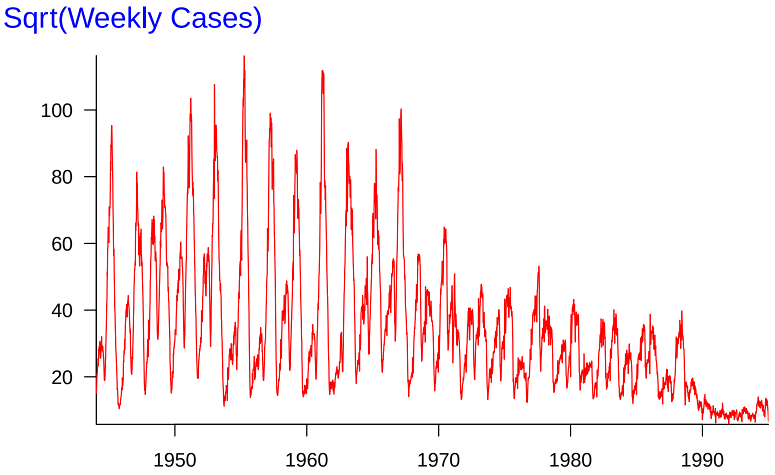
- Childhood diseases in New York City, 1928–1972
- Childhood diseases in Ontario, 1904–1989

Measles incidence in England and Wales, 1944–1995

Weekly Cases



Measles incidence in England and Wales, 1944–1995



Why study measles epidemics?

- $\sim 90,000$ children died from measles in 2016.
- A major cause of *vaccine-preventable* deaths.
- Potential impact in developed countries during vaccine scares (e.g., MMR scare in UK in 1990s).

- Understand past patterns
- Predict future patterns
- Manipulate future patterns
- Develop vaccination strategy that can. . .

Other reasons to model infectious disease epidemics

- Mathematical models make hypotheses and inferences precise
  - Give better advice to policymakers
  - Make better predictions
- Host-pathogen dynamics are important aspects of ecosystem dynamics
  - Infectious disease models more likely to be successful than predator-prey models
- Excellent data for human infectious diseases
  - Models can be tested!

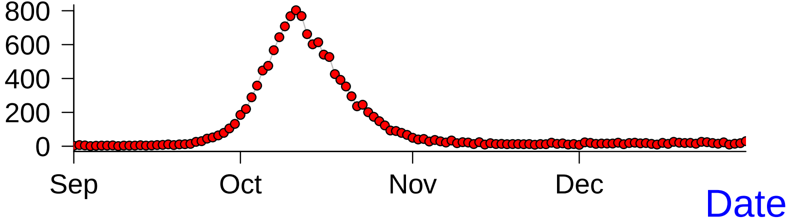
Modelling population dynamics childhood infections

- The basic SIR model cannot explain recurrent epidemics.
- What should we do?. . .
  1. Get depressed, drop the course.
  2. Keep developing models until we can explain recurrent epidemics.
- First, let's talk about tools that allow us to make our questions about time series data more precise.

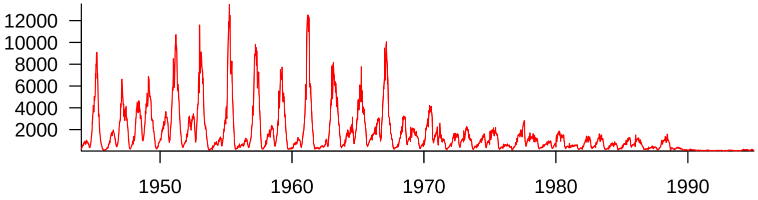
Epidemic Data Analysis

Time Plots of Temporal Epidemic Patterns

1918 P&I

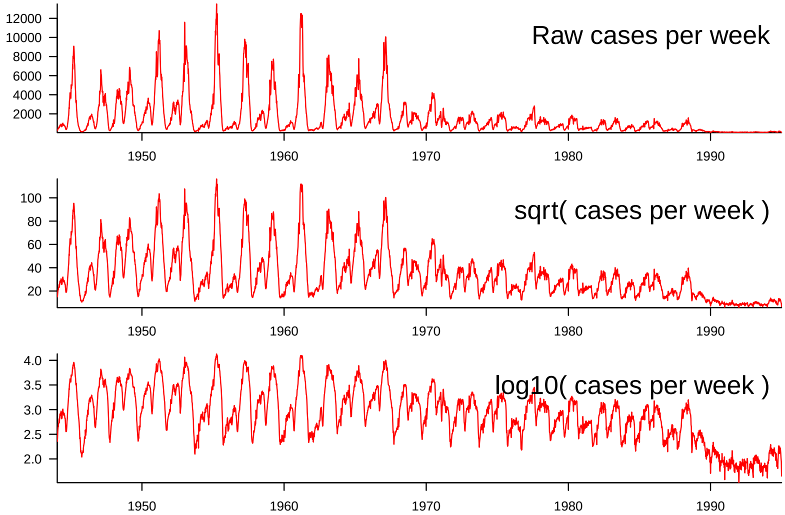


Weekly Measles in England and Wales



Time Plots of Transformed Data

- Reveal unobvious aspects of time series



Times Plots of Smoothed Data

- Reveal trends clouded by noise or seasonality
- *Moving Average:*

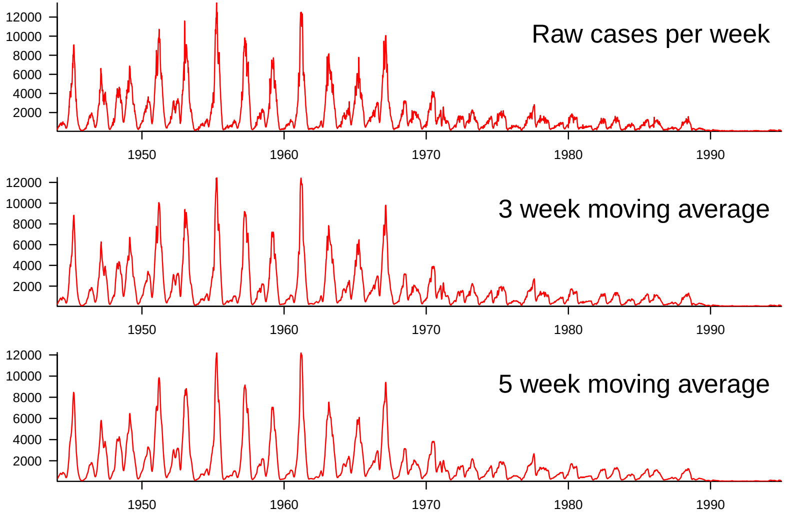
$$x_t \to \frac{1}{2a+1} \sum_{i=-a}^{a} x_{t+i}$$

  - Replace original data points x_t with averages of nearby points.
- *Linear filter:*

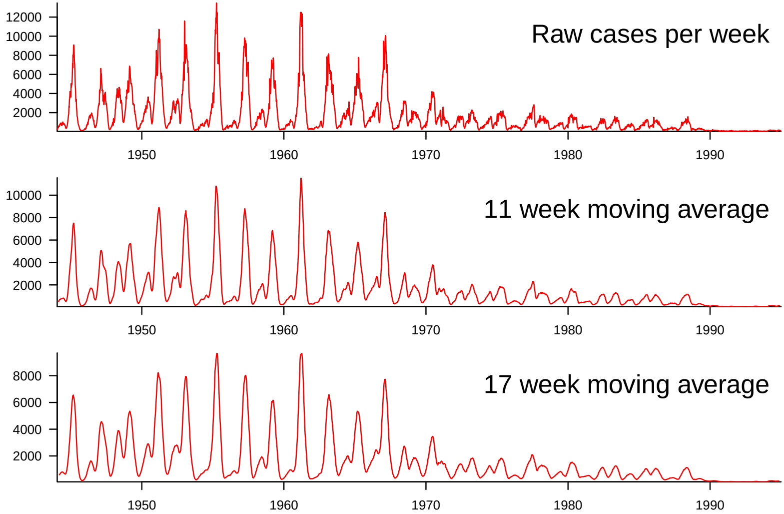
$$x_t \to \sum_{i=-\infty}^{\infty} \lambda_i x_{t+i}$$

  - Generalization of moving average.
  - *Weights* λ_i can be nonlinear functions of i.

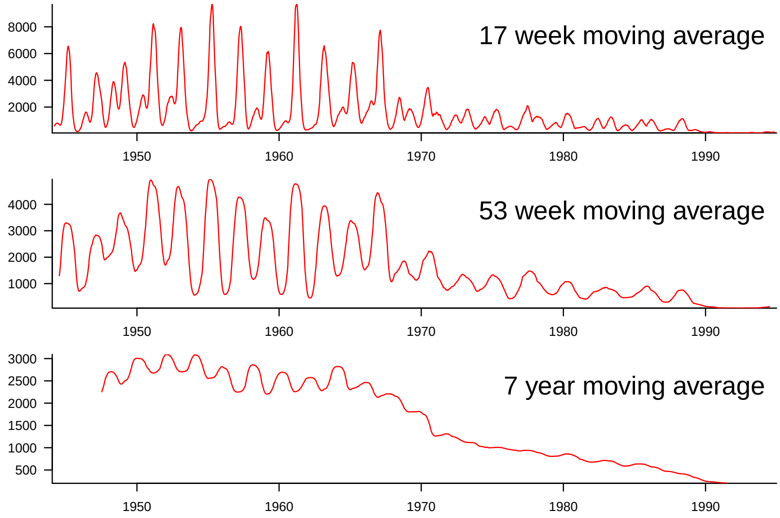
Times Plots of Smoothed Data



Times Plots of Smoothed Data



Times Plots of Smoothed Data



Correlation

- Recurrent epidemics $\Longrightarrow$ number of cases now is correlated with number of cases in the past and the future.
- Given N pairs of observations of different quantities, $\{(x_i, y_i) : i = 1, \dots, N\}$, the *correlation coefficient* is defined to be

$$r = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i - \bar{x})^2 \sum_{i=1}^{N}(y_i - \bar{y})^2}}$$

  where $\bar{x}$ and $\bar{y}$ are the means of $\{x_i\}$ and $\{y_i\}$, respectively.

Correlation

Properties of the correlation coefficient:

- $-1 \leq r \leq 1$ (Proof? Cauchy-Schwarz inequality)
- $r = 1 \iff$ all points lie on a line with positive slope ("complete positive correlation")
- $r = -1 \iff$ all points lie on a line with negative slope ("complete negative correlation")
- $r \simeq 0 \implies$ "uncorrelated"
- *Interpretation:* r^2 is the proportion of the variance in y explained by a linear function of x.

Derivations and discussions:

- MathWorld on r^2, Wikipedia on r^2
- Wikipedia on general coefficient of determination

Autocorrelation

- Given a single sequence of observations $\{x_t : t = 1, \ldots, N\}$, we can compute the correlation of each observation with the observation k time steps in the future.

- Thus, we consider the pairs of observations $\{(x_t, x_{k+t}) : t = 1, \ldots, N-k\}$ and define the *autocorrelation coefficient at lag k* to be

$$r_k = \frac{\sum_{t=1}^{N-k}(x_t - \bar{x}_{1,N-k})(x_{k+t} - \bar{x}_{k+1,N})}{\sqrt{\sum_{t=1}^{N-k}(x_t - \bar{x}_{1,N-k})^2 \sum_{t=1}^{N-k}(x_{k+t} - \bar{x}_{k+1,N})^2}}$$

where $\bar{x}_{1,N-k}$ and $\bar{x}_{k+1,N}$ are the means of first and last $N-k$ observations, respectively.

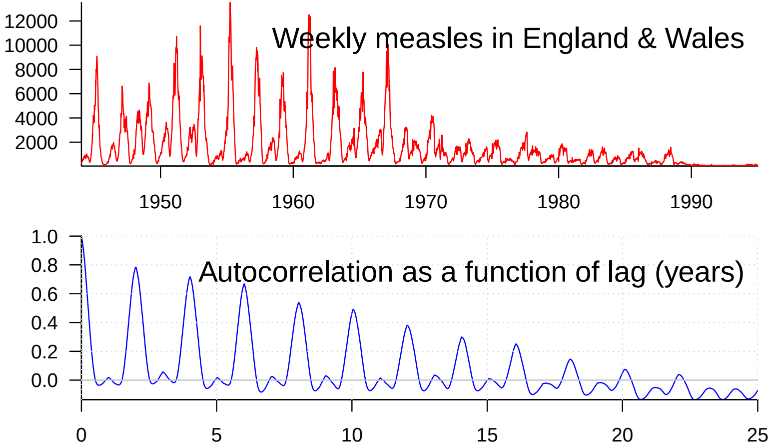
Autocorrelation

- If number of observations N is large and lag $k \ll N$ then

$$r_k \simeq \frac{\sum_{t=1}^{N-k}(x_t - \bar{x})(x_{k+t} - \bar{x})}{\sum_{t=1}^{N}(x_t - \bar{x})^2}$$

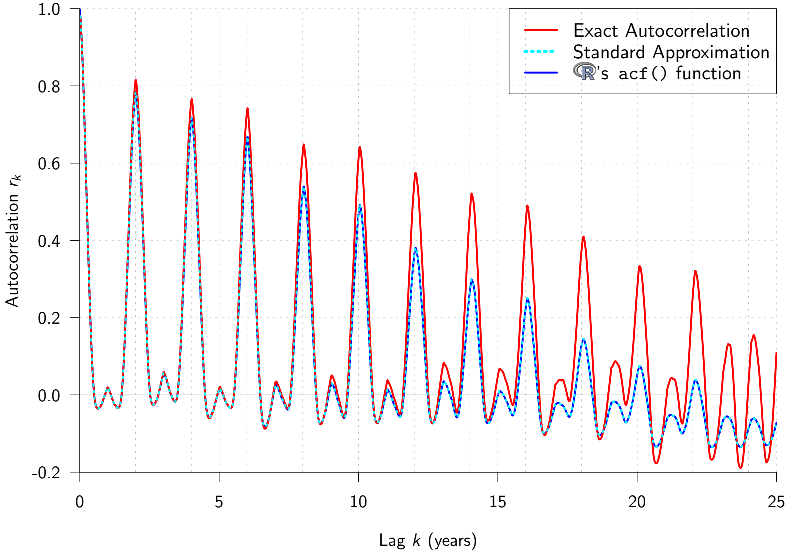
- Approximation of r_k is worse for larger lags k
- Plot of autocorrelation r_k as a function of lag k is called the *correlogram*.

Correlogram



- Peaks in correlogram $\Longrightarrow$ periodicities in original time series.
- Correlograms of temporal segments are often informative.

Correlogram: exact vs. approximate r_k



Spectral Density

- Can we compute the dominant periods in the time series? (Rather than estimating them by eye from the correlogram.)

- Express the time series as a Fourier series:

$$x_t = a_0 + \left(\sum_{p=1}^{(N/2)-1} \left(a_p \cos \omega_p t + b_p \sin \omega_p t \right) \right) + a_{N/2} \cos \pi t \,,$$

  where $\omega_p = 2\pi p/N$.

- Compute the *Fourier coefficients* $\{a_p\}$, $\{b_p\}$ by taking inner products with $\cos \omega_p t$ and $\sin \omega_p t$.

Spectral Density

- Fourier coefficients of x_t are:

$$a_0 = \bar{x} = \frac{1}{N}\sum_t x_t\,,$$

$$a_p = \frac{2}{N}\sum_t x_t \cos\omega_p t\,, \qquad b_p = \frac{2}{N}\sum_t x_t \sin\omega_p t\,,$$

$$a_{N/2} = \frac{1}{N}\sum_t (-1)^t x_t\,,$$

  where sum is over observation times.

- Estimated power spectral density (PSD) at frequency ω_p is*:

$$I(\omega_p) = \frac{N}{4\pi}\left(a_p^2 + b_p^2\right)$$

*The normalization by $N/4\pi$ is the convention chosen by Chatfield (2004, "Analysis of Time Series: An Introduction"). Other normalization conventions are also in common use.