TBB 4.3.1 True or False: If $S \subseteq \mathbb{R}$ and all points of S are isolated, then S is closed.

Sol. Could this be true? Example: $S = \mathbb{N}$.
Every point of S is isolated, and S has no accumulation points, so it must be closed.
(Alternatively, complement is union of open intervals $\Rightarrow$ open).

However, consider $S = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Every point of S is isolated.
However, 0 is an accumulation point of S, but $0 \notin S$. So S is not closed.
$\therefore$ Statement is FALSE.

TBB 4.2.1 / 4.35 Determine the ~~set of~~ interior points, accumulation points, isolated points, and boundary points for each set E. Which sets are open/closed/neither?

a) $\{1, 1/2, 1/3, 1/4, \dots\}$

b) $\{x \in \mathbb{R} : x^2 < 2\}$

c) $\{x \in \mathbb{Q} : x^2 < 2\}$

d) $\mathbb{R} \setminus \mathbb{N}$

<u>Sol.</u>

a) $E^\circ = \emptyset$, $E' = \{0\}$, $E_{isol} = E$, $\partial E = E \cup \{0\}$

E is <u>not open</u> since $E^\circ \neq E$.
E is <u>not closed</u> since 0 is an accumulation point but $0 \notin E$.

b) $E = \{x \in \mathbb{R} ; x^2 < 2\} = (-\sqrt{2}, \sqrt{2})$

$E^\circ = E$, $E' = [-\sqrt{2}, \sqrt{2}]$, $E_{isol} = \emptyset$, $\partial E = \{-\sqrt{2}, \sqrt{2}\}$

E is <u>open</u> since $E^\circ = E$.
E is <u>not closed</u> since $\sqrt{2}$ is an accumulation point but $\sqrt{2} \notin E$.

c) $E = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

$E^\circ = \emptyset$, $E' = [-\sqrt{2}, \sqrt{2}]$, $E_{isol} = \emptyset$, $\partial E = [-\sqrt{2}, \sqrt{2}]$

E is <u>not open</u> since $E^\circ \neq E$.
E is <u>not closed</u> since $\sqrt{2} \in E'$ but $\sqrt{2} \notin E$.

<u>Remark</u>: Later, we will consider metric spaces. If we view E a subset of $\mathbb{Q}$ with the standard topology (instead of $\mathbb{R}$), then E is actually a clopen set (i.e. both closed and open). In $\mathbb{R}$, the only clopen sets are $\emptyset$ and $\mathbb{R}$ (exercise).

d) $E = \mathbb{R} \setminus \mathbb{N}$ $\quad$ $\mathbb{N}$ is closed so $\mathbb{R} \setminus \mathbb{N}$ is open.

$E^\circ = E$, $E' = \mathbb{R}$, $E_{isol} = \emptyset$, $\partial E = \mathbb{N}$

E is <u>open</u> since $E^\circ = E$

E is <u>not closed</u> since $1 \in E'$ but $1 \notin E$.

FBB Thm 4.16 Let $A \subseteq \mathbb{R}$ and $B = \mathbb{R} \setminus A$. Show that A is open if and only if B is closed.

Sol. ($\Longrightarrow$) Suppose A is open, but B is not closed. Then there exists an accumulation point x of B with $x \notin B$. So $x \in A$, and since A is open, x is an interior point of A. So there is some interval $(x-\delta, x+\delta) \subseteq A$. This interval therefore contains no points of B. This contradicts the fact that x is an accumulation point of B. So B must be closed.

($\Longleftarrow$) Suppose B is closed, but A is not open. Then there exists a point $x \in A$ that is not an interior point of A. Hence, every interval $(x-\delta, x+\delta)$ must contain ~~poi~~ a point in $A^c = B$. Since $x \notin B$, this means that x must be an accumulation point of B. But B is closed, so actually $x \in B$ (i.e. $x \notin A$). This is a contradiction, so A must ~~be~~ be open.

<u>TBB 4.3.10</u> Show that $E^\circ = \text{int}(E)$ is an open set.

<u>Sol</u>. By definition, it suffices to show that every point of $\text{int}(E)$ is an interior point of $\text{int}(E)$, that is, $\text{int}(E) \subsetneq \text{int}(\text{int}(E))$.

So let $x \in \text{int}(E)$. Then there exists some open interval $(x-c, x+c) \subseteq E$. We want $(x-c, x+c) \subseteq \text{int}(E)$.

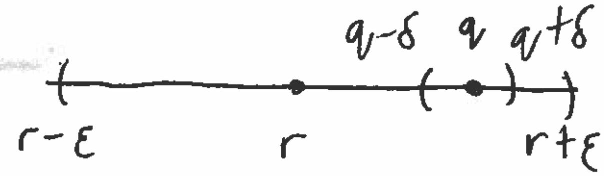
Since $(x-c, x+c)$ is open, ~~there exists~~ then for any arbitrary $y \in (x-c, x+c)$, there is an open interval $(y-d, y+d) \subseteq (x-c, x+c)$. But then $(y-d, y+d) \subseteq E$. So $y \in \text{int}(E)$. Since $y \in (x-c, x+c)$ is arbitrary, it follows that $(x-c, x+c) \subseteq \text{int}(E)$. Hence $x \in \text{int}(\text{int}(E))$. So $\text{int}(E) \subseteq \text{int}(\text{int}(E))$. $\blacksquare$

In fact $\text{int}(E) = \text{int}(\text{int}(E))$ since $\supseteq$ is always true.

TBB 4.2.16 Show that no set $S \subseteq \mathbb{R}$ has $S' = \mathbb{Q}$.

Sol. Claim: If every rational number is an accumulation point, every real number must be too.

Picture:



Let $r \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. We want to show that the interval $(r-\varepsilon, r+\varepsilon)$ contains an ~~rational number $q \in \mathbb{Q}$ with $q \neq$~~ element $s \in S$ with $s \neq r$.

By density, there exists $q \in \mathbb{Q}$ with $r < q < r+\varepsilon$. Let $\delta = \min\{q-r, r+\varepsilon-q\}$. Consider the open interval $(q-\delta, q+\delta)$. Since $q \in \mathbb{Q} = S'$, $(q-\delta, q+\delta)$ contains a point of $s \in S$. Our choice of δ implies $(q-\delta, q+\delta) \subseteq (r, r+\varepsilon)$, so $s \in (r, r+\varepsilon)$ (so $s \neq r$). It follows that $s \in (r-\varepsilon, r+\varepsilon)$ and since $s \neq r$, we conclude that r is an accumulation point of S.

So if $\mathbb{Q} \subseteq S'$, we must have $\mathbb{R} = S'$, so $\mathbb{Q} = S'$ is impossible.

TBB 4.2.9 True or False: For $S \subseteq \mathbb{R}$, every boundary point of S is an accumulation point of S.

Sol. FALSE. Consider $S = [0,1] \cup \{2\}$. 2 is an isolated point of S, which must also be a boundary point since every interval $(2-\varepsilon, 2+\varepsilon)$ for $\varepsilon > 0$ contains a point of S and a point of S^c. But an isolated point is never an accumulation point, and vice versa (proof: ex).

The statement is true if S contains no isolated points.

Sketch of proof:

$x \in \partial S$. 2 cases, $x \notin E$ or $x \in E$.
Each case implies every deleted neighborhood of x contains a point of E. Use the fact that x is isolated for case 2.