[BB #9.2.7 Let $f_n \to f$ pointwise on $[a,b]$. Which of the following are true?

~~a)~~

a) If each f_n is ~~increasing~~ nondecreasing, then so is f.

TRUE. Let $a \le x < y \le b$. Then $f_n(x) \le f_n(y)$ since ~~f~~ each f_n is nondecreasing.

But then $f(x) = \lim_{n\to\infty} f_n(x) \le \lim_{n\to\infty} f_n(y) = f(y)$,

so f is also non-decreasing.

b) If each f_n is discontinuous everywhere, then so is f.

FALSE. For each $x \in [a,b]$, define

$$f_n(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ \frac{1}{n}, & x \notin \mathbb{Q} \end{cases} \quad \text{for each } n \ge 1$$

Then each f_n is discontinuous everywhere.

However $\lim_{n\to\infty} f_n(x) = 0$ for each x, so

$f_n \to 0$ and $f \equiv 0$ is continuous. $\blacksquare$

B.S 8.1.1/8.1.11 Let $f_n(x) = \frac{x}{x+n}$ for all $n \geq 1$ and $x \geq 0$.

Does $\{f_n\}$ converge pointwise on $[0, \infty)$? If so, does it converge uniformly on $[0, a]$ and $[0, \infty)$?

Sol. We have $0 \leq \frac{x}{x+n} \leq \frac{x}{n}$ for all $n \geq 1$ and $x \geq 0$, so squeeze theorem $\Longrightarrow \lim_{n\to\infty} f_n(x) = 0$, so $\{f_n\}$ converges pointwise to $f \equiv 0$.

For $\{f_n\}$ to ~~be uniformly~~ converge uniformly, we must show that

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $\forall x \geq 0$, if $n \geq N$, then
$$|f_n(x) - 0| < \varepsilon.$$

Observe that $|f_n(x) - 0| = \left|\frac{x}{x+n}\right| = \frac{x}{x+n} \underset{\text{since } x \leq a}{\leq} \frac{a}{x+n} \underset{\text{since } x \geq 0}{\leq} \frac{a}{n}$.

So let $\varepsilon > 0$. Choose $N > \frac{a}{\varepsilon}$, and let $x \geq 0$ be arbitrary. If $n \geq N$, then
$$|f_n(x) - 0| \leq \frac{a}{n} \leq \frac{a}{N} < \varepsilon \text{ as desired.}$$

Note N ~~only~~ depends only on ε, not x.

Conclusion: $f_n \xrightarrow{\text{unif}} 0$ on $[0, a]$.

On $[0,\infty)$ we can't use previous argument to establish uniform convergence, since $[0,\infty)$ is unbounded.

<u>Claim</u>: f_n does not converge uniformly on $[0,\infty)$.

Need to show that $\exists \varepsilon > 0$ st. $\forall N \in \mathbb{N}$, $\exists x \geq 0$ ~~with~~ and $\exists n \geq N$ with $|f_n(x) - 0| \geq \varepsilon$.

Choose $\varepsilon = \underline{\frac{1}{2}}$. Let $N \in \mathbb{N}$ be arbitrary, and choose $x = \underline{N}$, and $n = \underline{N}$. Then

$$|f_n(x) - 0| = \left|\frac{x}{x+n}\right| = \frac{N}{N+N} = \frac{N}{2N} = \frac{1}{2} \geq \varepsilon.$$

So f_n is not unif. convergent on $[0,\infty)$. ■

<u>35 8.2.1</u> Let $f_n(x) = \dfrac{x^n}{1+x^n}$ for $n \geq 1$, $x \in [0,2]$.

Does f_n converge uniformly on $[0,2]$?

<u>Sol</u>. You could try to use the definitions, but there is an easier way. Recall that if f_n cont. & $f_n \xrightarrow{\text{unif}} f$ then f is continuous.

Here, the f_n are continuous. What is f?

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1/2 & x = 1 \\ 1 & 1 < x \leq 2 \end{cases}$$

But f is not continuous at $x = 1$! So the convergence cannot be uniform. ■

BS 9.4.1 ~~For each f_n, discuss convergence/ uniform~~

Discuss the ~~convergence~~/uniform convergence of the series $\sum_{k=1}^{\infty} f_k$ where $f_k(x)$ is

a) $f_k(x) = \dfrac{1}{k^2x^2}$, $x \neq 0$

Try M-test. Observe that $\dfrac{1}{k^2x^2} \leq \dfrac{1}{k^2}$

for $x^2 \geq 1$. Moreover, for any $a > 0$, if $x^2 \geq a^2$, then $\dfrac{1}{k^2x^2} \leq \dfrac{1}{k^2a^2}$

and $\sum_{k=1}^{\infty} \frac{1}{a^2} \cdot \frac{1}{k^2}$ converges (p-series).

So $\sum_{k=1}^{\infty} \frac{1}{k^2x^2}$ converges uniformly for $x^2 \geq a^2$

i.e. $|x| \geq a$. ie. on $(-\infty, a] \cup [a, \infty)$ for each $a > 0$.

~~However,~~ $\sum_{k=1}^{\infty} \frac{1}{k^2x^2}$ converges pointwise on $\mathbb{R} \setminus \{0\}$, but not uniformly.

Recall Cauchy's criterion: $\sum f_k$ unif conv. $\iff \forall \varepsilon > 0 \ \exists N^{\varepsilon}$ s.t. $\forall x \neq 0$, if $m > n \geq N$, then $|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$

The negation is $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists x_N \neq 0$ and $m > n \geq N$ with $|f_{n+1}(x) + \dots + f_m(x)| \geq \varepsilon$.

Choose $\varepsilon = 1$. Let N be arbitrary.
Choose $m = N+1$, $n = N$, and $x = \frac{1}{N+1}$.

Observe that $|f_{n+1}(x)| = |f_{N+1}(\frac{1}{N+1})| = |\frac{1}{(N+1)^2} \cdot (N+1)^2|$

$= 1 \geq \varepsilon$.

A4 Q4 Determine pointwise/~~con~~ uniform convergence of the series $\sum_{n\geq 1} \frac{x}{n(1+nx^2)}$.

Sol. Start with pointwise convergence. Fix $x \in \mathbb{R}$. If $x=0$, then series converges to 0 as each summand is 0.

So suppose $x \neq 0$. We have

$\frac{x}{n(1+nx^2)} \leq \frac{x}{n(nx^2)} = \frac{1}{x} \cdot \frac{1}{n^2}$. Note that

$\sum_{n\geq 1} \frac{1}{x} \cdot \frac{1}{n^2} = \frac{1}{x} \sum_{n\geq 1} \frac{1}{n^2}$ converges (to $\frac{1}{x} \cdot \frac{\pi^2}{6}$)

by p-series test with $p=2>1$. So by comparison test, $\sum_{n\geq 1} \frac{x}{n(1+nx^2)}$ converges for each $x \neq 0$ and hence converges pointwise on all of $\mathbb{R}$, ~~to the function~~

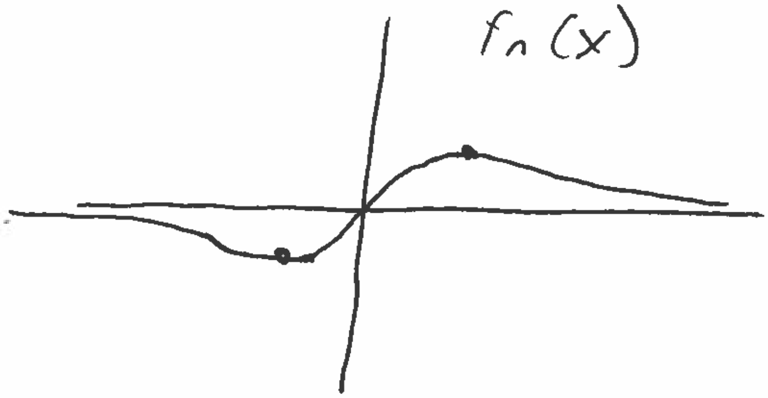
Now, is this series uniformly convergent on $\mathbb{R}$? If we want to use M-test, then for each $n \geq 1$, we want to bound the expression $|f_n(x)| = \left|\frac{x}{n(1+nx^2)}\right|$ by M_n which only depends on n. Lets try to determine global max/min of $f_n(x)$.

Note that $f_n(x)$ is (infinitely) differentiable:

$$f_n'(x) = \frac{n(1+nx^2) - 2n^2x^2}{n^2(1+nx^2)^2} = \frac{1-nx^2}{n(1+nx^2)^2}$$

and $f_n''(x) = \dfrac{2x(nx^2-3)}{(nx^2+1)^3}$

Local max/min must occur when $f_n' = 0$, in this case, will be global max/min



$f_n'(x) = 0 \Rightarrow 1 - nx^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{n}}$

Check that $f_n''(\frac{1}{\sqrt{n}}) < 0$ and $f_n''(-\frac{1}{\sqrt{n}}) > 0$ so by 2nd derivative test, get max of $f_n(\frac{1}{\sqrt{n}}) = \frac{1}{2n^{3/2}}$ and min of $f_n(-\frac{1}{\sqrt{n}}) = \frac{-1}{2n^{3/2}}$

So $|f_n(x)| \le \frac{1}{2n^{3/2}}$ for all x.

Now let $M_n = \frac{1}{2n^{3/2}}$. $\sum M_n$ converges by p-series, so $\sum f_n(x) = \sum \frac{x}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$. $\blacksquare$