Metric Spaces

A metric space (X,d) is a nonempty set with a metric $d: X \times X \to X$ such that for any points $x, y, z \in X$,

(1) $d(x,y) \geq 0$
(2) $d(x,y) = 0 \iff x = y$
(3) $d(x,y) = d(y,x)$
(4) $d(x,y) \leq d(x,z) + d(z,y)$.

Open ball is $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$
Closed ball is $B[x_0, r] = \{x \in X : d(x_0, x) \leq r\}$

IBB 13.5.8

True or false: In any metric space, $\overline{B(x_0,r)} = B[x_0,r]$.

Sol. False. Consider ~~any space~~ for example $\mathbb{R}$ with the discrete metric. $B(0,1) = \{0\}$ and $B[0,1] = \mathbb{R}$. Recall that

$$\overline{B(0,r)} = B(0,r) \cup \partial B(0,r).$$

What are the boundary points of $B(0,r)$?
If $a \in \mathbb{R}$ is a boundary point then certainly $B(a,1)$ should contain a point of $B(0,r)$ and a point of $B(0,r)^c$. But $B(a,1) = \{a\}$ only contains 1 point, so this is impossible. So the boundary is empty, and $\overline{B(0,r)} = B(0,r) \neq B[0,r]$.

Problem A Consider $(\mathbb{N}, d|_{\mathbb{N}})$ where d is the Euclidean metric on $\mathbb{R}$. Let $E \subseteq \mathbb{N}$ be nonempty. Find

a) E°

b) ∂E

c) $\overline{E}$

d) E_{isol}

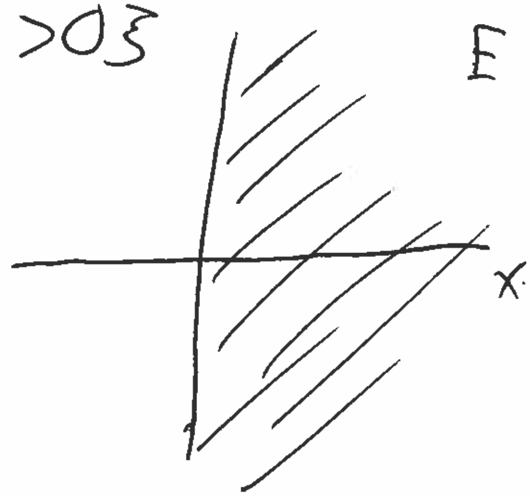
Sol. a) $E^\circ = E$. For any $x \in E$, consider $B(x,1) \subseteq E$.

b) $\partial E = \emptyset$. For any $a \in \mathbb{N}$, $B(a,1) = \{a\}$, so $B(a,1)$ does not contain a point of both E and E^c, so $a \notin \partial E$.

c) $\overline{E} = E$. We have $\overline{E} = E \cup \partial E = E \cup \emptyset = E$.

d) $E_{isol} = E$. For any point $a \in E$, $B(a,1) = \{a\}$.

Problem B Consider $\mathbb{R}^2$ with the discrete metric.
Let $E = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$



Find each of these sets:

a) E°

b) ∂E

c) $\bar{E}$

d) E_{isol}

e) E'

Sol. a) $E^\circ = E$. For any $x \in E$, $B(x,1) = \{x\} \subseteq E$.

b) $\partial E = \emptyset$. For any $x \in \mathbb{R}^2$, $B(x,1) = \{x\}$ does not contain points in E and E^c.

c) $\bar{E} = E$. $\bar{E} = E \cup \partial E = E$.

d) $E_{isol} = E$. For any $x \in E$, $B(x,1) = \{x\}$.

e) $E' = \emptyset$. For any $x \in \mathbb{R}^2$, $B(x,1) = \{x\}$, so $B(x,1) \setminus \{x\} = \emptyset$ does not contain a point of E.

Warmup: TBB 13.4.8 If $\{x_n\}$ and $\{y_n\}$ are ~~convergent~~ sequences in (X, d) and $\lim_{n\to\infty} d(x_n, y_n) = 0$, show that they are both convergent or both divergent.

Sol. Towards a contradiction, suppose WLOG that $\{x_n\}$ is convergent with $x_n \to x$. We will show that $\{y_n\}$ is also convergent with $y_n \to x$.

For any n, we have $d(x, y_n) \leq d(x, x_n) + d(x_n, y_n)$. $\star$

Since $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $x_n \to x$, then for any $\varepsilon > 0$, $\exists\, M_1, M_2 \in \mathbb{N}$ such that

note def of lim in $\mathbb{R}$ $\longrightarrow$ $|d(x_n, y_n) - 0| < \frac{\varepsilon}{2}$ for all $n \geq M_1$

and $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq M_2$

So let $K = \max\{M_1, M_2\}$. Then for all $n \geq N$ we have $d(x, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by $\star$.

TBB 13.4.7 If $\{x_n\}$ and $\{y_n\}$ are convergent sequeces in (X,d), show that $\lim_{n\to\infty} d(x_n,y_n)$ exists.

Sol. If $x_n \to x$ and $y_n \to y$, we expect that
$$\lim_{n\to\infty} d(x_n,y_n) = d(x,y).$$

So we want to show: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|d(x_n,y_n) - d(x,y)| < \varepsilon$ for all $n \geq N$.

We have
$$d(x_n,y_n) \leq d(x_n,x) + d(x,y_n)$$
$$\leq d(x_n,x) + d(y,y_n) + d(x,y)$$

so
$$d(x_n,y_n) - d(x,y) \leq d(x_n,x) + d(y,y_n)$$

On the other hand,
$$d(x,y) \leq d(x_n,x) + d(x_n,y)$$
$$\leq d(x_n,x) + d(y,y_n) + d(x_n,y_n)$$

So
$$d(x,y) - d(x_n,y_n) \leq d(x_n,x) + d(y,y_n).$$

It follows that $|d(x_n,y_n) - d(x,y)| \leq d(x_n,x) + d(y_n,y)$.

So let $\varepsilon > 0$ be given, and choose M_1, M_2 such that $d(x_n,x) < \frac{\varepsilon}{2}$ for all $n \geq M_1$ and $d(y_n,y) < \frac{\varepsilon}{2}$ for all $n \geq M_2$.

Then for all $n \geq N = \max\{M_1, M_2\}$, we have $|d(x_n,y_n) - d(x,y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ as desired.

IBB 13.S.13 Consider the set $C[0,1]$ with the two metrics:

$$D_1(f,g) = \max_{0 \le t \le 1} |f(t) - g(t)|$$

$$D_2(f,g) = \int_0^1 |f(t) - g(t)|\, dt.$$

Let B_1 be an open ball in $(C[0,1], D_1)$.
Let B_2 be an open ball in $(C[0,1], D_2)$.

a) Is B_1 open in $(C[0,1], D_2)$?
b) Is B_2 open in $(C[0,1], D_1)$?

Sol. The first question to ask: how do D_1 and D_2 relate to each other?

$$D_1(f,g) = \max_{0 \le t \le 1} |f(t) - g(t)| < r$$

$$\Rightarrow \max_{0 \le t \le 1} |f(t) - g(t)| = q \text{ for some } q < r \text{ (EVT)}$$

$$\Rightarrow D_2(f,g) = \int_0^1 |f(t) - g(t)| \le q(1-0) < r.$$

So for any $r > 0$, $B_{D_1}(f,r) \subseteq B_{D_2}(f,r)$. (A)

Is this useful for a) or b)?

Proof of b): Fix $f \in C[0,1]$, $r > 0$ and

$$B_2 := B_{D_2}(f,r) = \{g \in C[0,1] \mid D_2(f,g) < r\}$$

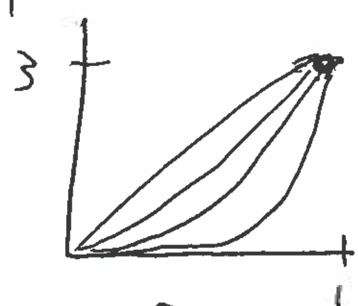
Since B_2 is open wrt D_2, (ex.) then each $g \in B_2$ is an interior point: $\exists\, s > 0$ st. $B_{D_2}(g,s) \subseteq B_2$

We must show that g is also an interior point with respect to D_1.

But $B_{D_1}(g,s) \subseteq B_{D_2}(g,s) \subseteq B_2$ by (*).
So B_2 is open in $(C[0,1], D_1)$.
What about a)? Consider the open ball
$B_1 = B_{D_1}(0,3)$. Claim: B_1 is not open wrt. D_2,
as $f \equiv 0$ is not an interior point, but $f \in B_1$.
It suffices to show that for every $r > 0$,
$B_{D_2}(0,r) \not\subseteq B_1$. So for each $r > 0$, need to
find $h \in B_{D_2}(0,r)$ with $h \notin B_1$.

Idea: Choose h such that $\max\limits_{0 \le t \le 1} |h(t) - 0| = 3$, but

$$\int_0^1 |h(t) - 0|\,dt < r.$$



area $\to 0$
as $r \to 0$

Let $h(t) = 3t^{3/r}$.
Observe that

$$\int_0^1 |h(t) - 0|\,dt = 3 \left. \frac{t^{3/r+1}}{3/r+1} \right|_0^1 = \frac{3}{\frac{3}{r}+1} < \frac{3}{\frac{3}{r}} = r$$

so $h \in B_{D_2}(0,r)$.

However, $\max\limits_{0 \le t \le 1} |h(t) - 0| = 3$ so $h \notin B_1 = B_{D_1}(0,3)$.

So it is not the case that B_1 is nec. open in $(C[0,1], D_2)$. ∎

TBB 13.4.3 Recall that $l_2(\mathbb{R})$ is the set of square-summable sequences:

$$l_2(\mathbb{R}) = \{(x_1, x_2, \ldots) \mid x_k \in \mathbb{R}, \sum_{k=1}^{\infty} x_k^2 < \infty\}.$$

If we define $d_2(x,y) = \left(\sum_{k=1}^{\infty} (x_k - y_k)^2\right)^{1/2}$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, then $(l_2(\mathbb{R}), d_2)$ is a metric space.

Q: Let $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots)$ be a sequence of points in $l_2(\mathbb{R})$ and $y = (y_1, y_2, \ldots) \in l_2(\mathbb{R})$. Is it true that

$$x^{(n)} \longrightarrow y \quad \text{in } l_2(\mathbb{R})$$

if and only if

$$\lim_{n\to\infty} x_k^{(n)} = y_k \quad \text{for each } k = 1, 2, \ldots ?$$

Sol. ($\Longrightarrow$) Suppose that $x^{(n)} \to y$. Then for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\sum_{k=1}^{\infty} (x_k^{(n)} - y_k)^2 < \varepsilon^2$ whenever $n \geq N$.

But for each fixed $k = 1, 2, \ldots$,

$$(x_k^{(n)} - y_k)^2 \leq \sum_{i=1}^{\infty} (x_i^{(n)} - y_i)^2 < \varepsilon^2 \text{ whenever } n \geq N$$

So $|x_k^{(n)} - y_k| = \sqrt{(x_k^{(n)} - y_k)^2} < \varepsilon$ whenever $n \geq N$, i.e., $\lim_{n\to\infty} x_k^{(n)} = y_k$.

($\Longleftarrow$) What about the converse?

Consider the case that $y = (0, 0, \dots)$.

Consider the sequence of points

$$x^{(n)} = (0, \dots, 0, \underset{\substack{\| \\ x_n^{(n)}}}{1}, 0, \dots)$$

where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots)$

and $x_k^{(n)} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$.

For each $k = 1, 2, \dots$, we have $\lim_{n \to \infty} x_k^{(n)} = 0$.

However, for any n, we have

$$\sqrt{\sum_{k=1}^{\infty} (x_k^{(n)} - y_k)^2} = \sqrt{\sum_{k=1}^{\infty} (x_k^{(n)})^2} = 1$$

So in particular, for $\varepsilon = 1/2$, there is no n such that $d_2(x^{(n)}, y) < \frac{1}{2}$, so $x^{(n)} \not\to y$, the converse must be false! $\blacksquare$

TBB 13.6.17 Consider the metric space $(C[0,1], d)$ where $d(f,g) = \int_0^1 |f-g|$.
Define the functional $T: C[0,1] \to \mathbb{R}$ by $T(f) = \int_0^1 f(t)dt$.
Is T continuous?

Sol. Yes. Let $f_n \to f$ in $(C[0,1], d)$, we must show that $T(f_n) \to T(f)$ in $(\mathbb{R}$, standard$)$.
Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$d(f_n, f) = \int_0^1 |f_n(t) - f(t)| dt < \varepsilon$$

But for any $n \in \mathbb{N}$, we have

$$\begin{aligned} |T(f_n) - T(f)| &= \left| \int_0^1 f_n(t)\,dt - \int_0^1 f(t)\,dt \right| \\ &= \left| \int_0^1 (f_n(t) - f(t))\,dt \right| \\ &\leq \int_0^1 |f_n(t) - f(t)|\,dt \quad \text{(exercise)} \end{aligned}$$

so $|T(f_n) - T(f)| \leq d(f_n, f) < \varepsilon$ for all $n \geq N$.
Hence T is continuous at f, and hence on $(C[0,1], d)$ since f is arbitrary. $\blacksquare$

TBB 13.5.4 Let E be a closed set in (X,d). Let $x \notin E$. Show that

$$\inf\{d(x,y) : y \in E\} > 0.$$

If E and F are disjoint closed sets, is it true that $\inf\{d(x,y) : x \in E, y \in F\} > 0$.

Sol. Since E is closed, x cannot be a limit point, as otherwise $x \in E$. So $\exists r > 0$ such that $B(x,r) \cap E = \emptyset$, as $x \notin E$. Hence for any $y \in E$, we must have $d(x,y) \geq r$. So

$$\inf\{d(x,y) : y \in E\} \geq r > 0 \text{ as desired.}$$

Answer is NO. Consider the metric space $\mathbb{R} \setminus \{0\}$ with the ~~usual~~ standard metric. Let $E = (-\infty, 0)$ and $F = (0, \infty)$. Then E and F are disjoint. Moreover E and F are closed as they contain their limit points (note $0 \notin \mathbb{R} \setminus \{0\}$). However, if we let $x_n = \frac{1}{n}$ and $y_n = \frac{-1}{n}$ for each $n \geq 1$, then $d(x_n, y_n) = \frac{2}{n}$, so since $\inf \leq \frac{2}{n}$ for all n, we have $\inf = 0$. ∎

TBB 13.3.5 Let $R[0,1]$ be the set of integrable functions on $[0,1]$. Let

$$d(f,g) = \int_0^1 |f(t)-g(t)|\,dt.$$

Is $(R[0,1], d)$ a metric space?

Sol. Non-negativity, symmetry are obviously satisfied. ~~d(f,g)~~ Triangle inequality also satisfied via triangle inequality on $\mathbb{R}$. We also need that $d(f,g)=0 \iff f=g$.

However, consider $f \equiv 0$

$$g(x) = \begin{cases} 1 & x = 1/2 \\ 0 & x \neq 1/2 \end{cases}$$

Both f and g are integrable and $|f(x)-g(x)| = g(x)$ for all $x \in [0,1]$.

But then $d(f,g) = \int_0^1 |f(t)-g(t)| = \int_0^1 g(t)\,dt = 0$ (using our knowledge of integrals).

However, $f \neq g$, so d cannot be a metric on $R[0,1]$. $\blacksquare$

TBB 13.5.15 (The Hilbert Cube)

Consider the following subsets of ℓ_2:

$H = \{(x_1, x_2, \ldots) \in \ell_2 \mid |x_i| \leq 1/i\}$

$G = \{(x_1, x_2, \ldots) \in \ell_2 \mid |x_i| < 1/i\}$

H is called the Hilbert cube.

a) Is H closed in ℓ_2?

b) Is G open in ℓ_2?

Sol. a) Let's show H is closed. Let $y = (y_1, y_2, \ldots)$ be a limit point of H. We must show $y \in H$. We know there exists a sequence of points $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots) \in H$ s.t. $x^{(n)} \xrightarrow[n\to\infty]{} y$.

From last week, we know this implies that for each k, $\lim_{n\to\infty} x_k^{(n)} = y_k$. So since each $x^{(n)} \in H$,

we have that for each k

$$|x_k^{(n)}| \leq 1/k \implies -\frac{1}{k} \leq x_k^{(n)} \leq \frac{1}{k}$$

$$\implies -\frac{1}{k} \leq \underbrace{\lim_{n\to\infty} x_k^{(n)}}_{=y_k} \leq \frac{1}{k} \quad \text{taking limits}$$

$$\implies -\frac{1}{k} \leq y_k \leq \frac{1}{k}$$

$$\implies |y_k| \leq 1/k$$

So $y \in H$ as desired.

b) Somewhat surprisingly, G is not open. As with last week, let $y := (0, 0, \ldots) \in G$. We will show that y is <u>not</u> an interior point of G. Recall that

$$B(0,r) = \{(x_1, x_2, \ldots) \in \ell_2 \mid \sqrt{\textstyle\sum_{i=1}^{\infty} x_i^2} < r\}$$

We need to show that for any $r > 0$, $B(0,r) \not\subseteq G$. In fact, it suffices to show that for all $0 < r < 1$, $B(0,r) \not\subseteq G$. In fact, it's enough to show that $B(0, \frac{1}{k}) \not\subseteq G$ for each $k \geq 2$.

Naive idea: let $x = (0, 0, \ldots, 0, \overset{x_k}{\frac{1}{k}}, \frac{1}{k^2}, \frac{1}{k^3}, \ldots)$ then $x \notin G$ since $|x_k| \not< \frac{1}{k}$. But sadly, $\sqrt{\sum_{i=1}^{\infty} x_i^2} = \sqrt{\sum_{i=1}^{\infty} (\frac{1}{k^2})^i}$ is not $< \frac{1}{k}$, so $x \notin B(0, \frac{1}{k})$.

New idea: let $x = (0, 0, \ldots, 0, \overset{x_{k^2}}{\frac{1}{k^2}}, (\frac{1}{k^2})^2, (\frac{1}{k^2})^3, \ldots)$

Now, we can compute

$$\sum_{i=1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} \left(\frac{1}{k^2}\right)^i = \sum_{i=0}^{\infty} \left(\frac{1}{k^2}\right)\left(\frac{1}{k^2}\right)^i$$

$$= \frac{\frac{1}{k^2}}{1 - \frac{1}{k^2}} = \frac{1}{k^2 - 1} < \frac{1}{k} \text{ since } k \geq 2$$

So $x \in B(0, \frac{1}{k})$, but as before $|x_{k^2}| \not< \frac{1}{k^2}$, so $x \notin G$. So G is not open, as $(0, 0, \ldots)$ is not an interior point. ∎

Compactness vs Completeness?

(X,d) metric space, $E \subseteq X$.

E is <u>compact</u> iff every open cover has a finite subcover

iff every sequence $(x_n) \subset E$ has a subsequence that converges to a point in E

~~E is complete~~

E is <u>complete</u> iff every Cauchy sequence $(x_n) \in E$ converges to a point of E.

<u>Fact</u>: E compact $\Longrightarrow$ E complete

<u>Proof</u>. Let $(x_n) \subset E$ be a Cauchy sequence. Since E is compact, we get a subsequence (x_{n_k}) of (x_n) with $(x_{n_k}) \to x \in E$. But since (x_n) is Cauchy, we also have $(x_n) \to x$ (exercise).

Since $x \in E$, it follows that E is complete. ■

<u>Fact</u>. E complete $\Longrightarrow$ E closed

<u>Proof</u>. Let $x \in E'$ and suppose $E \supset (x_n) \to x$. Need to show that $x \in E$. Since (x_n) converges, (x_n) is Cauchy. So for some $y \in E$, $(x_n) \to y$ as E is complete. But limits are unique, so $x = y \in E$ as desired. ■

TBB 13.9.3 Show that $f(x) = \cos x$ is a contraction mapping on $[1/2, 1]$ but not on $[0, \pi]$.

Sol. For $x \in [1/2, 1]$, we have $|f'(x)| = |-\sin x|$

$$= \sin x$$
$$\leq \sin(1)$$

Let $a, b \in [1/2, 1]$ with $a < b$. By MVT, there exists some $c \in (a,b)$ s.t.

$$\left| \frac{f(b) - f(a)}{b - a} \right| = \cancel{\phi} f'(c) \leq \sin(1)$$

Since this is true for any $a < b$ in $[1/2, 1]$, we have $|f(b) - f(a)| \leq \alpha |b - a|$ with $\alpha = \sin(1) \in [0, 1)$, so f is a contraction.

Now, for $x \in [0, \pi]$, we want to show that no $\alpha \in [0, 1)$ makes f a contraction.

Idea: show that you can always find $x, y \in [0, \pi]$ so that

$$\left| \frac{f(x) - f(y)}{x - y} \right|$$ is arbitrarily close to 1.

Fix $y = \frac{\pi}{2}$. then

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \left| \frac{\cos x}{x - \frac{\pi}{2}} \right|.$$

Observe that $\lim_{x \to \pi/2} \frac{\cos x}{x - \frac{\pi}{2}} \overset{H}{=} \lim_{x \to \pi/2} \frac{-\sin x}{1} = -1.$

So $\lim_{x \to \frac{\pi}{2}} \left| \frac{\cos x}{x - \frac{\pi}{2}} \right| = 1$ (abs. is continuous).

<u>So</u>: f is not contractive; if it was with constant $\alpha \in [0,1)$, then

$$\left| \frac{\cos x - 0}{x - \frac{\pi}{2}} \right| \leq \alpha \text{ for all } x \in [0, \pi]$$

implies that $1 = \lim_{x \to \frac{\pi}{2}} \left| \frac{\cos x}{x - \frac{\pi}{2}} \right| \leq \alpha < 1.$ $\times$

<u>BB 13.12.19</u> Let K be a compact subset of (X,d). Show that for any $x \in X$, there exists some $k \in K$ such that

$$d(k, x) = \inf \{ d(x,y) : y \in K \}.$$

Is the statement true if K is closed, but not compact?

Sol. Let $z = \inf\{d(x,y) : y \in K\} \in \mathbb{R}$. By properties of inf in $\mathbb{R}$, $\exists$ a sequence $(y_n) \in K$ such that

$$d(x, y_n) \xrightarrow[n\to\infty]{} z \quad (\bigstar).$$

Since K is compact (y_n) has a convergent subsequence (y_{n_K}) with $(y_{n_K}) \to w$, and $w \in K$.

Claim: $d(w,x) = z$, in which case we're done.

To see this, let's work with the subsequence. $(\bigstar)$ implies that $d(x, y_{n_K}) \xrightarrow[K\to\infty]{} z$.

For any $K \geq 1$, we have

$$|d(w,x) - z| = |d(w,x) - d(x,y_{n_K}) + d(x,y_{n_K}) - z|$$
$$\leq \underbrace{|d(w,x) - d(x,y_{n_K})|}_{= 1} + \underbrace{|d(x,y_{n_K}) - z|}_{= 2}$$

By Δ-inequality: $d(w,x) \leq d(w, y_{n_K}) + d(x, y_{n_K})$
$d(y_{n_K}, x) \leq d(w, y_{n_K}) + d(w, x)$

$\Longrightarrow$ 1 $\leq d(w, y_{n_K})$

So $|d(w,x) - z| \leq d(w, y_{n_K}) + |d(x, y_{n_K}) - z|$

Now let $\varepsilon > 0$ and choose M_1, M_2 such that

$$d(w, y_{n_k}) < \frac{\varepsilon}{2} \quad \text{for all } k \geq M_1$$
$$|d(x, y_{n_k}) - z| < \frac{\varepsilon}{2} \quad \text{for all } k \geq M_2.$$

Then for $k \geq \max\{M_1, M_2\}$, we have

$$\begin{aligned} |d(w,x) - z| &\leq d(w, y_{n_k}) + |d(x, y_{n_k}) - z| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

It follows that $|d(w,x) - z| = 0$, so $d(w,x) = z$ as desired. ∎

For the second part, consider $X = \{1, 1/2, 1/3, \ldots\} \cup \{-1\}$ with the ~~usual~~ standard ~~topo~~ metric inherited from $\mathbb{R}$.
Let $K = \{1, 1/2, 1/3, \ldots\}$. K is closed ~~the~~ because it has no limit points in X, and so it contains all of its limit points.
K is not compact, as $x_n = \frac{1}{n}$ is a sequence of points in K that does not have a subsequence that converges in K.
Choosing $x = -1$, note that $\inf\{d(-1, y) : y \in K\} = 1$ but there is no $k \in K$ such that $d(k, -1) = 1$