TA: Jeff

email: marshj16@mcmaster.ca

OH: 1PM-3PM Sunday, April 19 (HH 403)

LAST TUTORIAL TODAY

Question 1:

13.6.17 Let $\mathcal{C}[0,1]$ consist of the continuous functions on $[0,1]$ and furnished with the metric

$$d(f,g) = \int_0^1 |f(t) - g(t)|\, dt.$$

Define $T : \mathcal{C}[0,1] \to \mathbb{R}$ by

$$T(f) = \int_0^1 f(t)\, dt.$$

Is T continuous?

Solution: Yes! Assume $f_n \in C[0,1]$ then, $f \in C[0,1]$, and $f_n \xrightarrow{d} f$. Then

$$|T(f_n) - T(f)| = \left| \int_0^1 f_n(t)\,dt - \int_0^1 f(t)\,dt \right|$$

$$\leq \int_0^1 |f_n(t) - f(t)|\,dt = d(f_n, f).$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large s.t.

$n \geq N$ implies $d(f_n, f) < \varepsilon$. Then

$n \geq N$ implies $|T(f_n) - T(f)| \leq d(f_n, f) < \varepsilon$,

as desired.

Question 2:

13.6.16 Let $\mathcal{C}^1[a,b]$ consist of the continuously differentiable functions on $[a,b]$. Define for $f, g \in \mathcal{C}^1[a,b]$

$$d(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)| + \max_{a \leq t \leq b} |f'(t) - g'(t)|.$$

(a) Prove that d is a metric.

(b) Let $D : \mathcal{C}^1[a,b] \to \mathcal{C}[a,b]$ be defined by $D(f) = f'$. Prove that D is continuous. (Here, as usual, $\mathcal{C}[a,b]$ has the sup metric.)

Solution to a: Let $f, g, h \in C^1[a,b]$.

① $d(f,g) = 0 \iff f = g$.

$(\Rightarrow)$ $d(f,g) = 0 \Rightarrow \max_{a \leq t \leq b} |f(t) - g(t)| = 0$.

Let $t_0 \in [a,b]$. Then

$|f(t_0) - g(t_0)| \leq \max_{a \leq t \leq b} |f(t) - g(t)| = 0$. So, $f(t_0) = g(t_0)$.

Since $t_0 \in [a,b]$ was arbitrary, $f=g$.

(1) If $f=g$, then $d(f,g) = d(f,f)$

$$= \max_{a \le t \le b} |f(t) - f(t)| + \max_{a \le t \le b} |f'(t) - f'(t)|$$

$$= \max_{a \le t \le b} \{0\} + \max_{a \le t \le b} \{0\} = 0 + 0 = 0.$$

2 $d(f,g) = d(g,f)$.

Indeed, $d(f,g) = \max_{a \le t \le b} |f(t) - g(t)| + \max_{a \le t \le b} |f'(t) - g'(t)|$

$$= \max_{a \le t \le b} |g(t) - f(t)| + \max_{a \le t \le b} |g'(t) - f'(t)| = d(g,f).$$

3 $d(f,g) \geq 0$.

Indeed, $d(f,g) = \underbrace{\max_{a \le t \le b} |f(t) - g(t)|}_{\geq 0} + \underbrace{\max_{a \le t \le b} |f'(t) - g'(t)|}_{\geq 0}$

$$\geq 0.$$

(4) $d(f,g) \le d(f,h) + d(h,g)$.

Indeed, $d(f,g) = \max_{a\le t\le b} |f(t)-g(t)| + \max_{a\le t\le b} |f'(t)-g'(t)|$

$\le \max_{a\le t\le b} \{|f(t)-h(t)|+|h(t)-g(t)|\}$

$+ \max_{a\le t\le b} \{|f'(t)-h'(t)|+|h'(t)-g'(t)|\}$

$\le \max_{a\le t\le b} |f(t)-h(t)| + \max_{a\le t\le b} |h(t)-g(t)|$

$+ \max_{a\le t\le b} |f'(t)-h'(t)| + \max_{a\le t\le b} |h'(t)-g'(t)|$

$= \max_{a\le t\le b} |f(t)-h(t)| + \max_{a\le t\le b} |f'(t)-h'(t)|$

$+ \max_{a\le t\le b} |h(t)-g(t)| + \max_{a\le t\le b} |h'(t)-g'(t)|$

$= d(f,h) + d(h,g)$.

<u>Solution to (b)</u>: Assume $f_n \in C^1[a,b]$ $\forall n\in\mathbb{N}$, $f \in C^1[a,b]$, and $f_n \xrightarrow{d} f$. Then

$$\|D(f_n) - D(f)\| = \|f_n' - f'\|$$

$$= \sup_{a \leq t \leq b} |f_n'(t) - f'(t)|$$

$$= \max_{a \leq t \leq b} |f_n'(t) - f'(t)| \qquad (EVT)$$

$$\leq \max_{a \leq t \leq b} |f_n(t) - f(t)| + \max_{a \leq t \leq b} |f_n'(t) - f'(t)|$$

$$= d(f_n, f).$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large s.t. $n \geq N$ implies $d(f_n, f) < \varepsilon$. Then $n \geq N$ implies $\|D(f_n) - D(f)\| \leq d(f_n, f) < \varepsilon$, as desired.

Question 3: Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \cos(\cos(x))$$

for all $x \in \mathbb{R}$. Show that the equation

$x = \cos(\cos(x))$ has a <u>unique</u> solution.

<u>Solution</u>: Want to apply Contraction Mapping Principle. Domain and codomain of f are both complete (wrt the standard metric), so it suffices to verify that f is a <u>contraction</u> (i.e. $|f(x) - f(y)| \le |x-y| \ \forall x, y \in \mathbb{R}$).

Mean Value theorem?

$\exists\, t \in (x,y)$ s.t. $f'(t) = \dfrac{f(x)-f(y)}{x-y}$

$\Rightarrow \sin(t)\sin(\cos(t)) = \dfrac{f(x)-f(y)}{x-y}$

$\Rightarrow |\sin(t)|\,|\sin(\cos(t))| = \dfrac{|f(x)-f(y)|}{|x-y|} \qquad (*)$

<u>Note</u>: $|\sin(\cos(t))| \neq 1$.

Indeed, $|\sin(\cos(t))| = 1$

$\Rightarrow \cos(t) \in \left\{\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}+\pi, -\frac{\pi}{2}-\pi, \frac{\pi}{2}+2\pi, \dots\right\}$

$\Rightarrow |\cos(t)| \geq \frac{\pi}{2}$

$\Rightarrow |\cos(t)| \not\leq 1$

$\Rightarrow$ Contradiction!

So, $|\sin(\cos(t))| < 1 \quad \forall t \in \mathbb{R}$.

Since $|\sin(\cos(t))|$ is 2π-periodic in t and continuous, Extreme Value THM

it achieves its sup at some $t_0 \in [-\pi, \pi]$. Let

$\lambda = |\sin(\cos(t_0))|$. Then $|\sin(\cos(t))| \leq \lambda$

Note: $0 < \lambda < 1$.

$\forall t \in \mathbb{R}$. So, referring to $(*)$,

$$\underbrace{|\sin(t)| \cdot}_{\leq 1} \lambda \geq \frac{|f(x) - f(y)|}{|x-y|}$$

$$\Rightarrow \quad \lambda \geq \frac{|f(x) - f(y)|}{|x-y|}$$

$$\Rightarrow \quad |f(x) - f(y)| \leq \lambda |x - y|$$

$\Rightarrow$ f is a contraction.

So, by contraction mapping THM,

$\exists ! \; z_0 \in \mathbb{R}$ s.t. $z_0 = f(t_0)$.

i.e., t_0 is the unique solution to the equation $t_0 = \cos(\cos(t_0))$.

Question 4: An evil wizard has trapped you. In order to escape, you must show that a series of the form $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on some domain $D \subseteq \mathbb{R}$.

Which theorem should you use?

(Choose one.)

A. Intermediate Value Theorem

B. Contraction Mapping Principle

C. p-series converge $(p > 1)$

D. Weierstrass M-test

<u>Question 5</u>: The wizard is enraged by your unanimous (and correct) answer. He challenges you again. In order to defeat the wizard, you must show that an equation of the form

$$f(x) - x = 0$$

has a unique solution. Which theorem should you use?

(Choose one.)

A) Intermediate Value Theorem

B) Mean Value Theorem

C) Contraction Mapping Principle

D) Weierstrass M-test

Question 6: The wizard, defeated, falls to his knees. His fake beard falls off and you realize that the wizard is actually your best friend, John, in disguise. He was just pulling a harmless prank.

What have you done?

In order to atone, you must ace the following true or false

Quiz.

TRUE or FALSE

f differentiable $\Rightarrow$ f continuous

<u>Question 7</u>: TRUE or FALSE

f continuous $\Rightarrow$ f differentiable

<u>Question 8</u>: TRUE or FALSE

If $f_n \xrightarrow{\text{uniformly}} f$, then $f_n' \xrightarrow{\text{uniformly}} f'$

(assuming f_n and f are differentiable).

<u>Question 9</u>: TRUE or FALSE

Continuous on $[a,b]$ $\Rightarrow$ Integrable on $[a,b]$

Question 10 : TRUE or FALSE

Integrable on [a,b] $\Rightarrow$ Continuous on [a,b]

Question 11 : TRUE or FALSE

Complete $\Rightarrow$ Compact

Question 12 : TRUE or FALSE

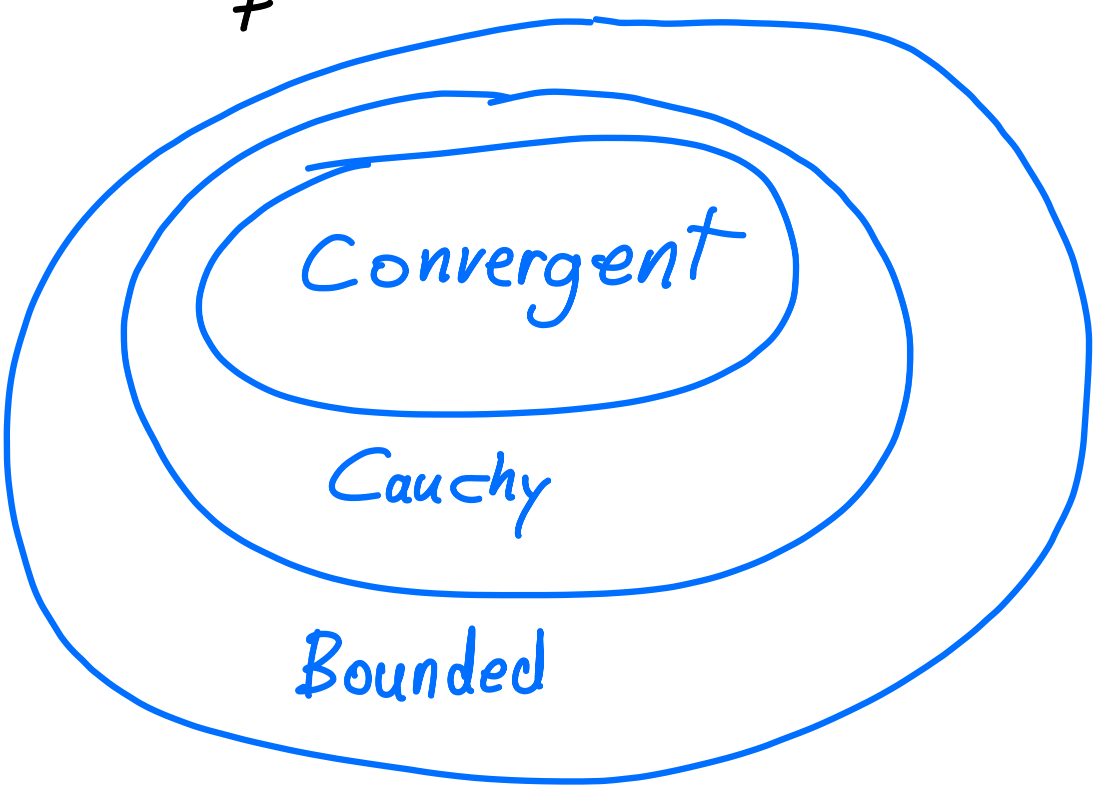
Compact $\Rightarrow$ Complete

Question 13 : TRUE or FALSE

Every convergent sequence in every metric space converges

Congratulations! You have atoned for your misdoings against John. Unfortunately, he has already blocked you.

<u>Question 14</u>: Working in the category of general metric spaces, draw a Venn diagram with the categories "convergent sequences," "Cauchy sequences," and "bounded sequences."



Question 15: TRUE or FALSE

For any map $f : \mathbb{R} \to \mathbb{R}$ and any real

numbers $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$, we have

$$\left| \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \right| \leq \sum_{i=1}^{n} |f(x_i)| (x_i - x_{i-1}).$$

Question 16: Using the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, dx$$

on $C[0,1]$, is the set

$$X = \{ f \in C[0,1] \mid |f(x)| < 1 \ \forall x \in [0,1] \}$$

compact?

Solution: No! Let $f_n(x) = 1 - \frac{1}{n}$ $\forall n \in \mathbb{N}$ and $x \in [0,1]$.

Then $f_n \xrightarrow{d} f$, where $f(x) = 1 \ \forall x \in [0,1]$.

But clearly $f \notin X$, so X is not closed

and therefore not complete.

Question 17: TRUE or FALSE

$$\sup_{x \in D} |f(x)| \geq \left| \sup_{x \in D} f(x) \right|$$

Question 18: TRUE or FALSE

$$\inf_{x \in D} |f(x)| \leq \left| \inf_{x \in D} f(x) \right|$$

Question 19: TRUE or FALSE

Contractions may be discontinuous.

Question 20:

13.9.3 Show that the function $f(x) = \cos x$ is a contraction mapping on $[1/2, 1]$ but is not a contraction map on $[0, \pi]$.

Solution: Let $x, y \in [1/2, 1]$. Then

by the MVT, $\exists\, z \in [x, y]$ s.t. $f'(z) = \dfrac{f(x) - f(y)}{x - y}$.

i.e., $$\sin(z) = \frac{\cos(x) - \cos(y)}{x - y}$$

But LHS $\neq 1$ $\forall\, z \in [1/2, 1]$.

However, sin is increasing on $[1/2, 1]$,

so $$\frac{|\cos(x) - \cos(y)|}{|x - y|} \leq \sin(1)$$

$$\Rightarrow |\cos(x) - \cos(y)| = \sin(1)\,|x - y|.$$

i.e., cos is a contraction on $[1/2, 1]$ with constant $\lambda = \sin(1)$.

Next we show that cos is <u>not</u> a contraction on $[0,\pi]$.

i.e. we show that for all $\lambda \in (0,1)$, $\exists x_0, y_0 \in [0,\pi]$ s.t. $|\cos(x_0)-\cos(y_0)| > \lambda |x_0 - y_0|$.

Let $\lambda \in (0,1)$. Let $x_0 = \frac{\pi}{2}$. Note that

$$\lim_{y_0 \to \frac{\pi}{2}} \frac{\cos(\frac{\pi}{2}) - \cos(y_0)}{\frac{\pi}{2} - y_0} = -\sin\left(\frac{\pi}{2}\right) = -1$$

$$\Rightarrow \lim_{y_0 \to \frac{\pi}{2}} \frac{|\cos(\frac{\pi}{2}) - \cos(y_0)|}{|\frac{\pi}{2} - y_0|} = 1$$

$\Rightarrow$ For $y_0 \neq \frac{\pi}{2}$ sufficiently close to $\frac{\pi}{2}$, we have $\left| \frac{|\cos(\frac{\pi}{2}) - \cos(y_0)|}{|\frac{\pi}{2} - y_0|} - 1 \right| < 1 - \lambda$

$$\Leftrightarrow -(1-\lambda) < \frac{|\cos(\frac{\pi}{2})-\cos(y_0)|}{|\frac{\pi}{2}-y_0|} - 1 < 1-\lambda$$

$$\Rightarrow \lambda < \frac{|\cos(\frac{\pi}{2})-\cos(y_0)|}{|\frac{\pi}{2}-y_0|}$$

$$\Rightarrow \lambda|\tfrac{\pi}{2}-y_0| < |\cos(\tfrac{\pi}{2})-\cos(y_0)|,$$

as desired.

Question 21:

13.12.1 Show that every closed subset of a compact set is also compact.

Solution: Let M be a compact metric space and $A \subseteq M$ a closed subset. Let $\{U_\alpha \mid \alpha \in I\}$ be an open covering of A. Let $V = M \setminus A$. Then V is open and $\{U_\alpha \mid \alpha \in I\} \cup \{V\}$ is an open covering of M. Since M is compact, it admits a finite subcover. V may or may not belong to this subcover. In any case, this subcover also covers A. If V belongs to the subcover throw V away! You are left with a finite subset of $\{U_\alpha \mid \alpha \in I\}$ which covers A, thereby proving that A is compact.

Question 22: Fill in the blank...

For a metric space M, M is covering compact if and only if M is sequentially compact.

Question 23:

13.12.8 Show that the unit ~~sphere~~ ball in $\mathcal{C}[a,b]$, that is, the set

$$\{f \in \mathcal{C}[a,b] : |f(x)| \leq 1, \quad x \in [a,b]\}$$

is not compact.

Solution: We are using the metric

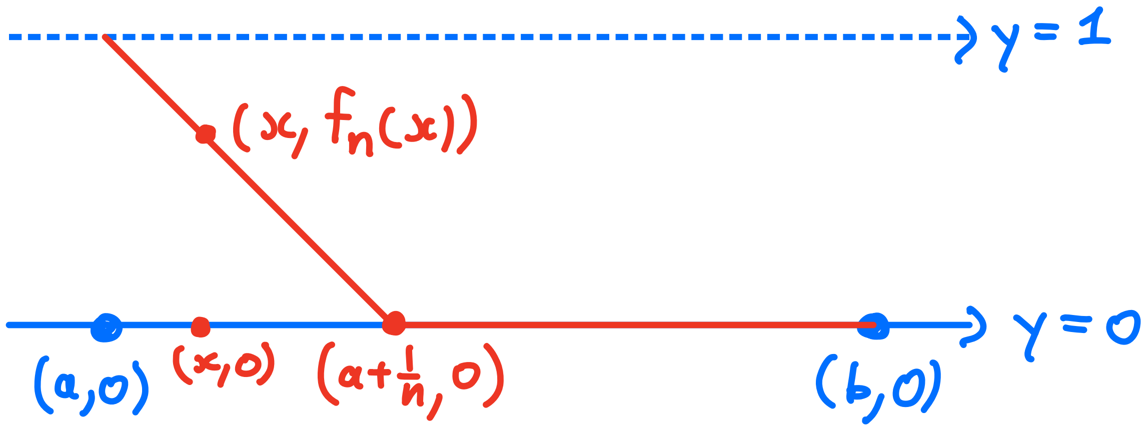
$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)| \quad \forall f,g \in C[0,1].$$

Recall: Compact $\Rightarrow$ sequentially compact.

So it suffices to argue that

$$X = \{f \in C[a,b] \mid |f(x)| \leq 1 \ \forall x \in [a,b]\}$$

is not sequentially compact. Let $f_n : [a,b] \longrightarrow \mathbb{R}$.



Then f_n is a sequence in X with no subsequential limit in X. Indeed, every subsequence of f_n must converge pointwise to the non-continuous function $f(x) = \begin{cases} 1, & x=a \\ 0, & \text{else} \end{cases}$. Therefore, a subsequence of f_n which converges wrt d (i.e. converges uniformly) must also converge to this f. But each f_n is continuous and the uniform limit of continuous functions is again continuous — contradicting the non-continuity of f!!!

Question 24:

13.12.9 Show that if K is a compact subset of a metric space (X, d), then for any $x \in X$ there is a point $k \in K$ so that

$$d(k, x) = \inf\{d(x, y) : y \in K\}.$$

Show that if K is not compact, but merely closed, this would not necessarily be true. If K is complete but not compact, is this always true?

See Note 359

Solution: Let $y_0, y_1, y_2, \dots \in K$ be a sequence such that $\inf\{d(x,y) \mid y \in K\} = \lim_{n\to\infty} d(x, y_n)$ (basic property of inf). Compact $\Rightarrow$ sequentially compact, so $\exists$ a subsequence y_{k_n} converging to some $y' \in K$. Then

$$\inf\{d(x,y) \mid y \in K\} = \lim_{n\to\infty} d(x, y_n)$$

$$= \lim_{n\to\infty} d(x, y_{k_n}) \quad \left(\begin{array}{l}\text{subsequence of convergent sequence}\\ \text{converges to same limit}\end{array}\right)$$

$$= d(x, \lim_{n\to\infty} y_{k_n}) \quad (\text{continuity of metrics})$$

$$= d(x, y), \text{ as desired.}$$

Next: Show that K closed is not enough.

Take the metric space $M = \mathbb{R} \setminus \{0\}$, the point $x = -1 \in M$, and the closed subset $K = (0, \infty) \subseteq \mathbb{R}$. We use the standard metric $d(x,y) = |x-y| \ \forall x, y \in M$.

Then $\inf\{d(-1,y) \mid y \in (0,\infty)\} = 1$,

but $d(-1,y) > 1 \ \forall y \in (0,\infty)$.

<u>Next</u>: Show that K complete is not enough.

Take the metric space

$$M = \ell_\infty = \left\{ \begin{array}{l} \text{sequences } x_0, x_1, x_2, \ldots \in \mathbb{R} \text{ s.t.} \\ \sup_n |x_n| < \infty \end{array} \right\}$$

with the sup metric:

$$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sup_n |x_n - y_n|.$$

Let $x = (0,0,0,\ldots) \in \ell_\infty$, and take the subset $K = \{(1,0,0,\ldots), (0,\frac{1}{2},0,\ldots), (0,0,\frac{1}{3},\ldots), \ldots\}$.

To see that K is a complete metric

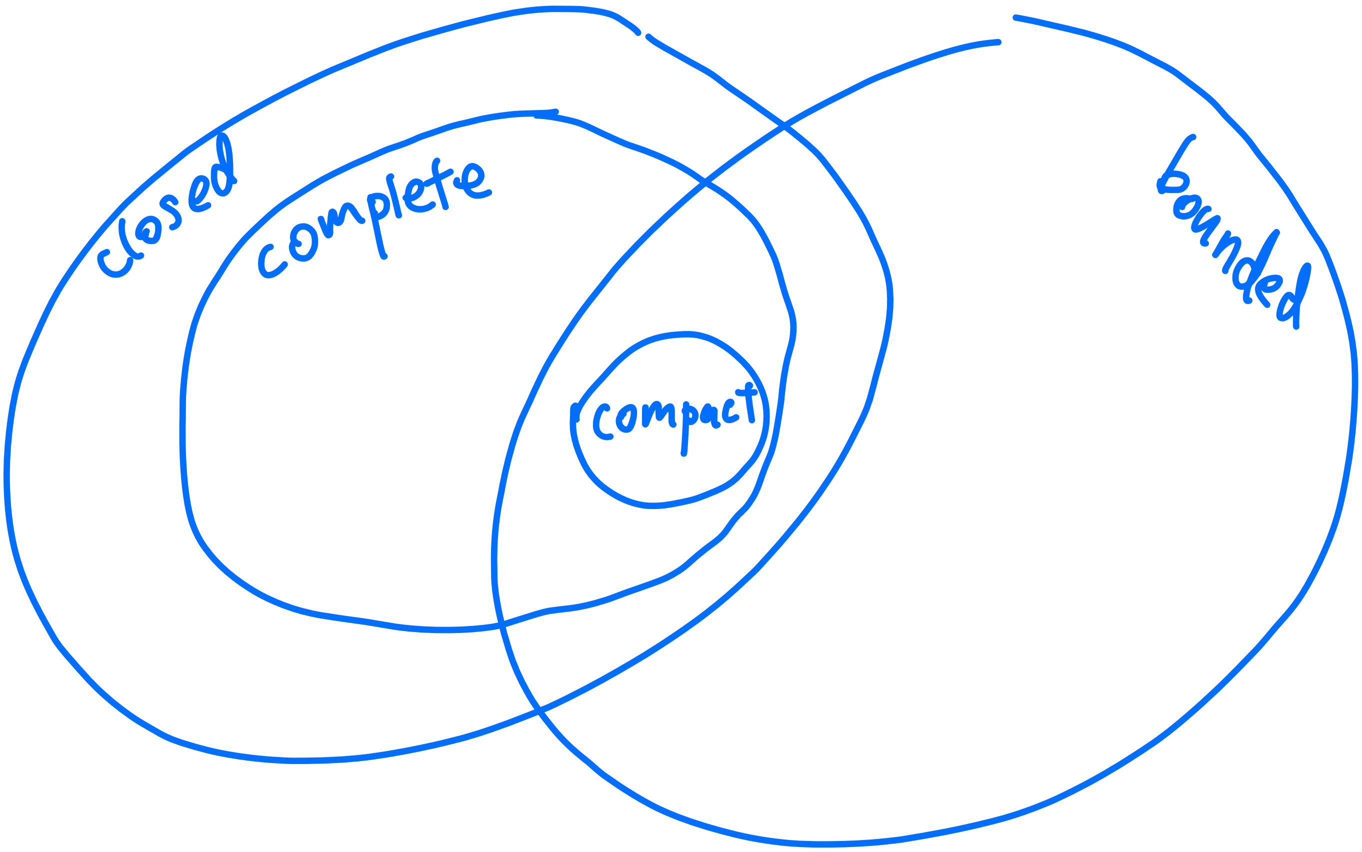
space (wrt the sup metric), note that every point in K is an isolated point.

So, every Cauchy sequence in K is eventually constant, and therefore convergent.

Now, $\inf\{d(x,y) \mid y \in K\} = 0$, but $d(x,y) > 0 \ \forall y \in K$.

<u>Question 25</u>: The continuous image of a compact set is... compact!!!

<u>Question 26</u>: Make a Venn diagram with the categories "complete," "closed," "bounded," "compact."



Advice for the exam:

Get practice with:

- Compact sets
- Contractions / the Contraction Mapping Principle
- The Mean Value Theorem
- Proving a map of metric spaces is continuous

↳ Proving things about integrals

FROM THE DEFINITION

Question 27: Let $M=\mathbb{R}^2$ with the metric

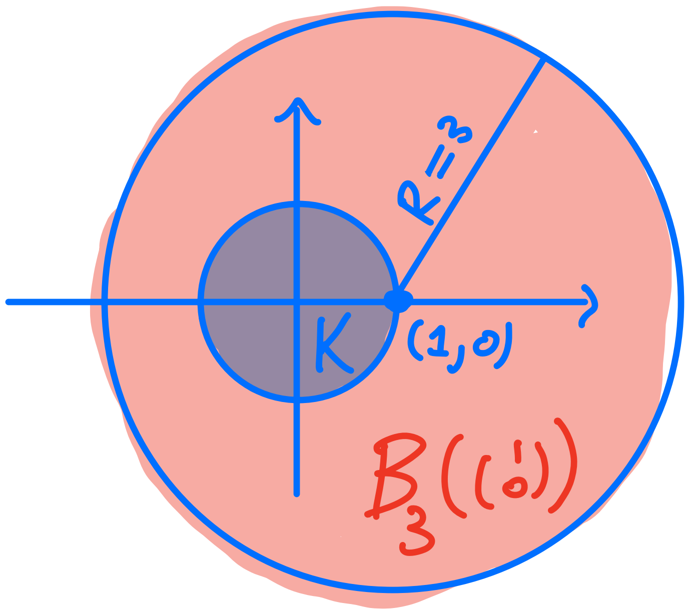
$$d\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \sqrt{(x_0-x_1)^2+(y_0-y_1)^2}.$$

Let $K=\left\{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_0^2+y_0^2 \leq 1\right\}$.

Is K compact? Consider the open cover $\left\{B_\varepsilon\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \;\middle|\; \varepsilon>0\right\}$ of K. Find a finite subcover.

Solution: K is compact by Heine-Borel THM.

$\left\{B_3\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right\}$ is a finite subcover.



$R=3$
K
$(1,0)$
$B_3\left(\binom{1}{0}\right)$