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LAST TUTORIAL TODAY LHH 403

Question 1:

13.6.17 Let $\mathcal{C}[0, 1]$ consist of the continuous functions on $[0, 1]$ and furnished with the metric

$$d(f, g) = \int_0^1 |f(t) - g(t)| dt.$$

Define $T : \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ by

$$T(f) = \int_0^1 f(t) dt.$$

Is T continuous?

Question 2:

13.6.16 Let $\mathcal{C}^1[a, b]$ consist of the continuously differentiable functions on $[a, b]$. Define for $f, g \in \mathcal{C}^1[a, b]$

$$d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)| + \max_{a \leq t \leq b} |f'(t) - g'(t)|.$$

- (a) Prove that d is a metric.
- (b) Let $D : \mathcal{C}^1[a, b] \rightarrow \mathcal{C}[a, b]$ be defined by $D(f) = f'$. Prove that D is continuous. (Here, as usual, $\mathcal{C}[a, b]$ has the sup metric.)

Question 3: Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \cos(\cos(x))$$

for all $x \in \mathbb{R}$. Show that the equation

$x = \cos(\cos(x))$ has a unique solution.

Question 4: An evil wizard has trapped you.

In order to escape, you must show that

a series of the form $\sum_{k=0}^{\infty} f_k(x)$ converges

uniformly on some domain $D \subseteq \mathbb{R}$.

Which theorem should you use?

(Choose one.)

A. Intermediate Value Theorem

B. Contraction Mapping Principle

C. p -series converge ($p > 1$)

D. Weierstrass M-test

Question 5: The wizard is enraged by your unanimous (and correct) answer. He challenges you again. In order to defeat the wizard, you must show that an equation of the form

$$f(x) - x = 0$$

has a unique solution. Which theorem should you use?

(Choose one.)

A) Intermediate Value Theorem

B) Mean Value Theorem

C) Contraction Mapping Principle

D) Weierstrass M-test

Question 6: The wizard, defeated, falls to his knees. His fake beard falls off and you realize that the wizard is actually your best friend, John, in disguise. He was just pulling a harmless prank.

What have you done?

In order to atone, you must ace the following true or false

Quiz.

TRUE or FALSE

f differentiable $\Rightarrow f$ continuous

Question 7: TRUE or FALSE

f continuous $\Rightarrow f$ differentiable

Question 8: TRUE or FALSE

If $f_n \xrightarrow{\text{uniformly}} f$, then $f_n' \xrightarrow{\text{uniformly}} f'$

(assuming f_n and f are differentiable).

Question 9: TRUE or FALSE

Continuous \Leftrightarrow Integrable
on $[a, b]$ on $[a, b]$

Question 10: TRUE or FALSE

Integrable \Rightarrow Continuous
on $[a, b]$ on $[a, b]$

Question 11: TRUE or FALSE

Complete \Rightarrow Compact

Question 12: TRUE or FALSE

Compact \Rightarrow Complete

Question 13: TRUE or FALSE

Every convergent
sequence in every
metric space
converges

Congratulations! You have atoned for

your misdoings against John. Unfortunately,

he has already blocked you.

Question 14: Working in ~~the category of~~

general metric spaces, draw a Venn

diagram with the categories "convergent

sequences," "Cauchy sequences," and

"bounded sequences."

Question 15: TRUE or FALSE

For any map $f: \mathbb{R} \rightarrow \mathbb{R}$ and any real numbers $x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n$, we have

$$\left| \sum_{i=1}^n f(x_i)(x_i - x_{i-1}) \right| \leq \sum_{i=1}^n |f(x_i)| (x_i - x_{i-1}).$$

Question 16: Using the metric

$$d(f, g) = \int_0^1 |f(x) - g(x)| dx$$

on $C[0, 1]$, is the set

$$X = \left\{ f \in C[0, 1] \mid |f(x)| < 1 \quad \forall x \in [0, 1] \right\}$$

compact?

Question 17: TRUE or FALSE

$$\sup_{x \in D} |f(x)| \geq \left| \sup_{x \in D} f(x) \right|$$

Question 18: TRUE or FALSE

$$\inf_{x \in D} |f(x)| \leq \left| \inf_{x \in D} f(x) \right|$$

Question 19: TRUE or FALSE

Contractions may be
discontinuous.

Question 20:

13.9.3 Show that the function $f(x) = \cos x$ is a contraction mapping on $[1/2, 1]$ but is not a contraction map on $[0, \pi]$.

... prove that $\cos(x) = x$ has a unique solution $x \in [\frac{1}{2}, 1]$.

Question 21:

13.12.1 Show that every closed subset of a compact set is also compact.

Question 22: Fill in the blank...

For a metric space M , M is covering

compact _____ M is

sequentially compact.

Question 23:

13.12.8 Show that the unit ~~sphere~~^{ball} in $C[a, b]$, that is, the set
$$\{f \in C[a, b] : |f(x)| \leq 1, x \in [a, b]\}$$

is not compact. (Use sup metric.)

Question 24:

13.12.9 Show that if K is a compact subset of a metric space (X, d) , then for any $x \in X$ there is a point $k \in K$ so that

$$d(k, x) = \inf\{d(x, y) : y \in K\}.$$

Show that if K is not compact, but merely closed, this would not necessarily be true. If K is complete but not compact, is this always true?

SEE NOTE 359

Question 25: The continuous image of a compact set is...

Question 26: Make a Venn diagram with the categories "complete," "closed," "bounded," "compact."

Advice for the exam:

Get practice with:

↳ Compact sets

↳ Contractions / the Contraction Mapping Principle

↳ The Mean Value Theorem

↳ Proving a map of metric spaces is continuous

↳ Proving things about integrals

FROM THE DEFINITION

Question 27: Let $M = \mathbb{R}^2$ with the metric

$$d\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

Let $K = \left\{ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2 \mid x_0^2 + y_0^2 \leq 1 \right\}$.

Is K compact? Consider the open

cover $\left\{ B_\varepsilon\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \mid \varepsilon > 0 \right\}$

of K . Find a finite subcover.