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OH: 1PM–3PM, April 19 (Sunday)

LAST TUTORIAL TODAY LOHH 403

Question 1:

13.6.17 Let $\mathcal{C}[0,1]$ consist of the continuous functions on $[0,1]$ and furnished with the metric

$$d(f,g) = \int_0^1 |f(t) - g(t)|\,dt.$$

Define $T : \mathcal{C}[0,1] \to \mathbb{R}$ by

d_0

$$T(f) = \int_0^1 f(t)\,dt.$$

Is T continuous?

Solution: Yes! Let $f_n \in C[0,1]$ with

$f_n \xrightarrow{d} f \in C[0,1]$.

WTS: $\lim_{n\to\infty} d_0(T(f_n), T(f)) = 0$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large s.t. $n \geq N$ implies $d(f_n, f) < \varepsilon$. Let $n \geq N$. Then

$$d_0(T(f_n), T(f)) = |T(f_n) - T(f)|$$

$$= \left|\int_0^1 f_n(x)\,dx - \int_0^1 f(x)\,dx\right|$$

$$= \left|\int_0^1 (f_n(x) - f(x))\,dx\right| \leq \int_0^1 |f_n(x) - f(x)|\,dx$$

$$= d(f_n, f) < \varepsilon.$$

Question 2:

13.6.16 Let $\mathcal{C}^1[a, b]$ consist of the continuously differentiable functions on $[a, b]$. Define for $f, g \in \mathcal{C}^1[a, b]$

$$d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)| + \max_{a \leq t \leq b} |f'(t) - g'(t)|.$$

(a) Prove that d is a metric.

(b) Let $D : \mathcal{C}^1[a, b] \to \mathcal{C}[a, b]$ be defined by $D(f) = f'$. Prove that D is continuous. (Here, as usual, $\mathcal{C}[a, b]$ has the sup metric.) $\to d_0$

Solution to (b): Let $f_n \in C^1[a,b]$ with

$$f_n \xrightarrow{d} f \in C^1[a,b]$$

WTS: $\lim_{n\to\infty} d_0(D(f_n), D(f)) = 0.$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large s.t. $n \geq N$ implies $d(f_n, f) < \varepsilon$. Let $n \geq N$. Then,

$$d_0(\mathcal{D}(f_n), \mathcal{D}(f)) = \sup_x |\mathcal{D}(f_n) - \mathcal{D}(f)|$$

$$= \sup_x |f_n'(x) - f'(x)|$$

$$= \max_x |f_n'(x) - f'(x)| \quad (EVT)$$

$$\leq \underbrace{\max_x |f_n(x) - f(x)|}_{\geq 0} + \max_x |f_n'(x) - f'(x)|$$

$$= d(f_n, f) < \varepsilon.$$

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Question 3: Define $f: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \cos(\cos(x))$$

for all $x \in \mathbb{R}$. Show that the equation

$x = \cos(\cos(x))$ has a unique solution.

Question 4: An evil wizard has trapped you. In order to escape, you must show that a series of the form $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on some domain $D \subseteq \mathbb{R}$.

Which theorem should you use?

(Choose one.)

A. Intermediate Value Theorem

B. Contraction Mapping Principle

C. p-series converge $(p>1)$

D. Weierstrass M-test

Question 5: The wizard is enraged by your unanimous (and correct) answer. He challenges you again. In order to defeat the wizard, you must show that an equation of the form

$$f(x) - x = 0$$

has a unique solution. Which theorem should you use?

(Choose one.)

A) Intermediate Value Theorem

B) Mean Value Theorem

C) Contraction Mapping Principle

D) Weierstrass M-test

<u>Question 6</u>: The wizard, defeated, falls to his knees. His fake beard falls off and you realize that the wizard is actually your best friend, John, in disguise. He was just pulling a harmless prank.

What have you done?

In order to atone, you must ace the following true or false

Quiz.

TRUE or FALSE

f differentiable $\Rightarrow$ f continuous

Question 7: TRUE or FALSE

f continuous $\Rightarrow$ f differentiable

Question 8: TRUE or FALSE

If $f_n \xrightarrow{\text{uniformly}} f$, then $f_n' \xrightarrow{\text{uniformly}} f'$

(assuming f_n and f are differentiable).

Question 9: TRUE or FALSE

Continuous on $[a,b]$ $\Rightarrow$ Integrable on $[a,b]$

Question 10 : TRUE or FALSE

Integrable on $[a,b]$ $\Rightarrow$ Continuous on $[a,b]$

Question 11 : TRUE or FALSE

Complete $\Rightarrow$ Compact

Question 12 : TRUE or FALSE

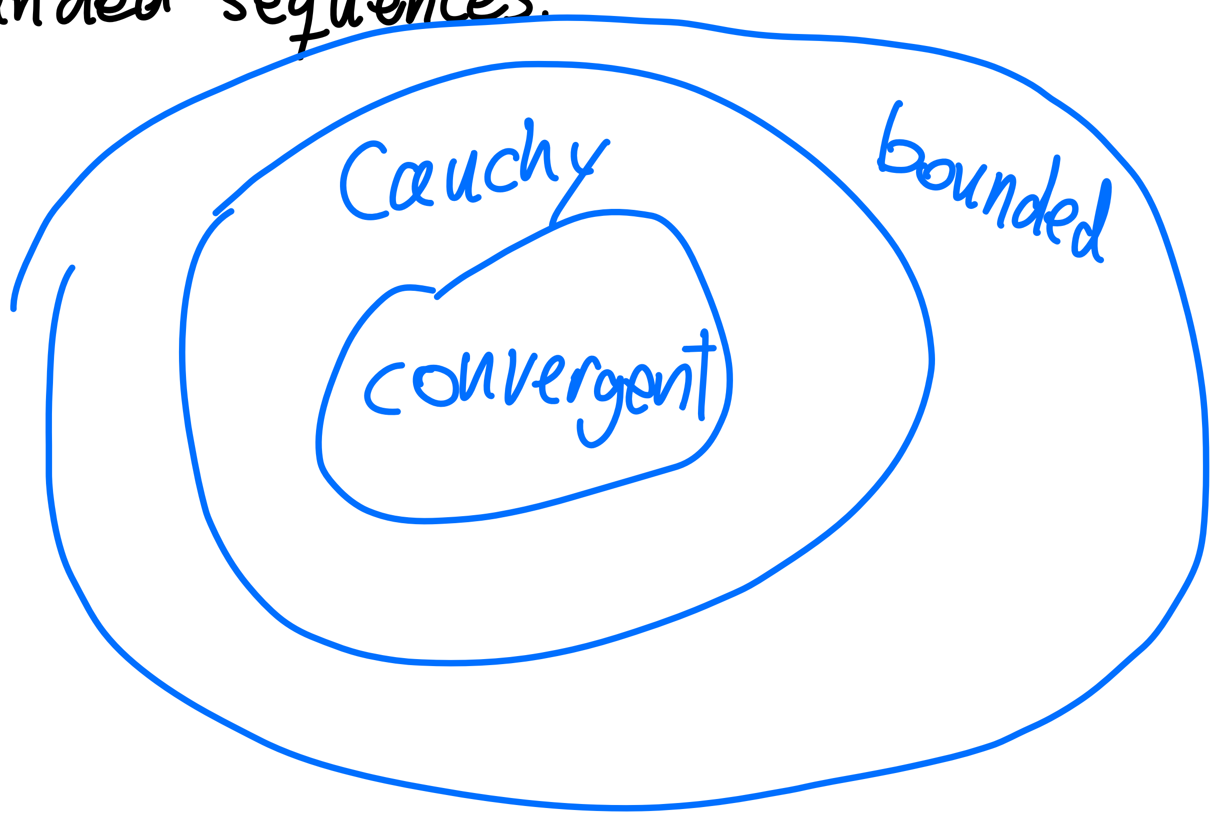
Compact $\Rightarrow$ Complete

Question 13: TRUE or FALSE

Every convergent sequence in every metric space converges

Congratulations! You have atoned for your misdoings against John. Unfortunately, he has already blocked you.

<u>Question 14</u>: Working in ~~the category of~~ general metric spaces, draw a Venn diagram with the categories "convergent sequences," "Cauchy sequences," and "bounded sequences."



Question 15: TRUE or FALSE

For any map $f: \mathbb{R} \to \mathbb{R}$ and any real numbers $x_0 \le x_1 \le x_2 \le \cdots \le x_n$, we have

$$\left| \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \right| \le \sum_{i=1}^{n} |f(x_i)| (x_i - x_{i-1}).$$

Question 16: Using the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)| \, dx$$

on $C[0,1]$, is the set

$$X = \{ f \in C[0,1] \mid |f(x)| < 1 \ \forall x \in [0,1] \}$$

compact?

Solution: NO! (because not closed).

$f_n(x) = 1 - \frac{1}{n} \quad \forall x \in [0,1]$, then

$f_n \xrightarrow{d} f \quad (f(x) = 1 \quad \forall x)$,

but $f \notin X$.

<u>Question 17</u>: TRUE or FALSE

$$\sup_{x \in D} |f(x)| \geq \left| \sup_{x \in D} f(x) \right|$$

<u>Question 18</u>: TRUE or FALSE

$$\inf_{x \in D} |f(x)| \leq \left| \inf_{x \in D} f(x) \right|$$

<u>Question 19</u>: TRUE or FALSE

Contractions may be discontinuous.

Question 20:

13.9.3 Show that the function $f(x) = \cos x$ is a contraction mapping on $[1/2, 1]$ but is not a contraction map on $[0, \pi]$.

... prove that $\cos(x) = x$ has a unique solution $x \in [\frac{1}{2}, 1]$.

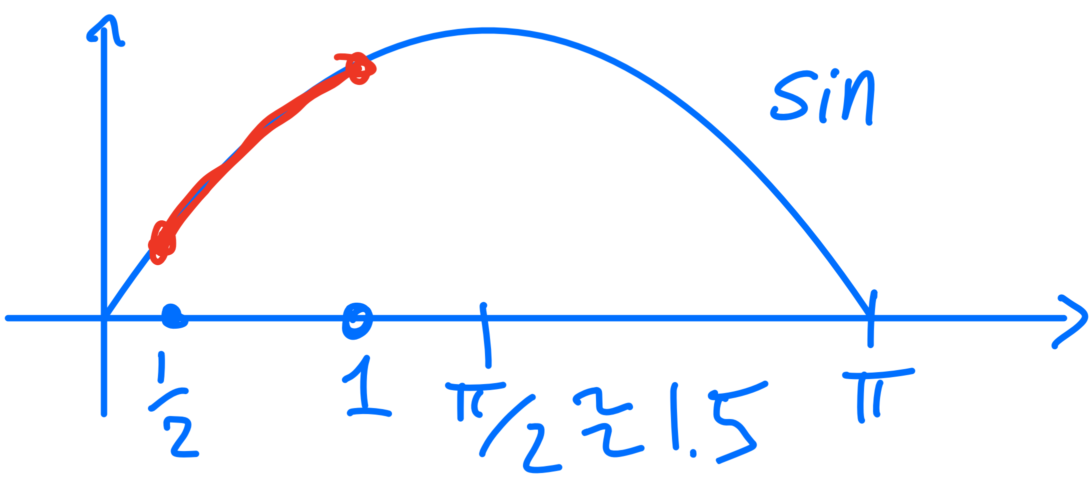
Solution: Mean Value THM? Let $\frac{1}{2} \leq x < y \leq 1$.

By MVT, $\exists t \in [x, y]$ s.t. $\dfrac{f(x) - f(y)}{x - y} = f'(t)$.

$$\Leftrightarrow \frac{\cos(x) - \cos(y)}{x - y} = -\sin(t).$$

$$\Rightarrow \frac{|\cos(x) - \cos(y)|}{|x - y|} = |\sin(t)|.$$

But sin is increasing on $[\frac{1}{2}, 1]$.



$$\frac{|\cos(x) - \cos(y)|}{|x - y|} = |\sin(t)| = \sin(t) \leq \sin(1).$$

$$|\cos(x)-\cos(y)| \le \underbrace{\sin(1)}_{<1}|x-y|.$$

CONCLUSION: cos is a contraction on $[\frac{1}{2}, 1]$ with constant $\sin(1)$.

Now to see cos is not a contraction on $[0,\pi]$.

WTS: $\forall \lambda \in (0,1), \exists x_0, y_0 \in [0,\pi]$ s.t.

$$|\cos(x_0)-\cos(y_0)| > \lambda |x_0 - y_0|.$$

Let $\lambda \in (0,1)$. Let $x_0 = \frac{\pi}{2}$. Then

$$\lim_{y_0 \to \frac{\pi}{2}} \frac{\cos(\frac{\pi}{2})-\cos(y_0)}{\frac{\pi}{2}-y_0} = -\sin(\frac{\pi}{2}) = -1$$

derivative

$$\Rightarrow \lim_{y_0 \to \frac{\pi}{2}} \frac{|\cos(\frac{\pi}{2}) - \cos(y_0)|}{|\frac{\pi}{2} - y_0|} = 1.$$

Salient point: For $y_0 \neq \frac{\pi}{2}$ close to $\frac{\pi}{2}$,

we have $\left| \frac{|\cos(\frac{\pi}{2}) - \cos(y_0)|}{|\frac{\pi}{2} - y_0|} - 1 \right| < 1 - \lambda.$

$$\Rightarrow \frac{|\cos(\frac{\pi}{2}) - \cos(y_0)|}{|\frac{\pi}{2} - y_0|} > \lambda \Rightarrow |\cos(\tfrac{\pi}{2}) - \cos(y_0)| > \lambda |\tfrac{\pi}{2} - y_0|.$$

Question 21:

13.12.1 Show that every closed subset of a compact set is also compact.

Question 22: Fill in the blank...

For a metric space M, M is covering compact ________________ M is sequentially compact.

Question 23:

13.12.8 Show that the unit ball ~~sphere~~ in $\mathcal{C}[a,b]$, that is, the set

$$\{f \in \mathcal{C}[a,b] : |f(x)| \leq 1, \quad x \in [a,b]\}$$

is not compact.

$$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

Question 24:

13.12.9 Show that if K is a compact subset of a metric space (X, d), then for any $x \in X$ there is a point $k \in K$ so that

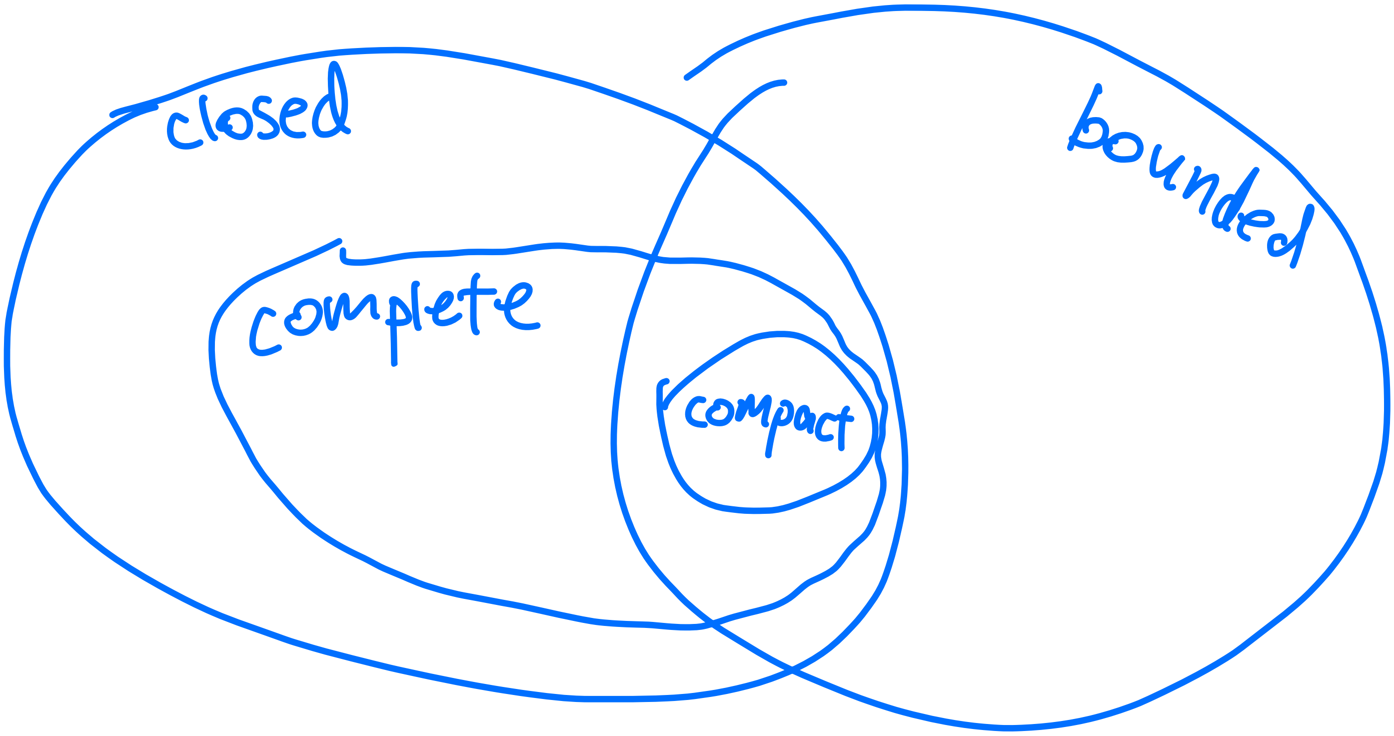
$$d(k, x) = \inf\{d(x, y) : y \in K\}.$$

Show that if K is not compact, but merely closed, this would not necessarily be true. If K is complete but not compact, is this always true?

SEE NOTE 359

<u>Question 25</u>: The continuous image of a compact set is... compact.

<u>Question 26</u>: Make a Venn diagram with the categories "complete," "closed," "bounded," "compact."



Advice for the exam:

Get practice with:

- Compact sets
- Contractions / the Contraction Mapping Principle
- The Mean Value Theorem
- Proving a map of metric spaces is continuous

↳ Proving things about integrals

FROM THE DEFINITION

Question 27: Let $M=\mathbb{R}^2$ with the metric

$$d\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \sqrt{(x_0-x_1)^2+(y_0-y_1)^2}.$$

Let $K=\left\{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_0^2+y_0^2 \leq 1\right\}.$

Is K compact? Consider the open cover $\left\{B_\varepsilon\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \;\middle|\; \varepsilon>0\right\}$ of K. Find a finite subcover.