

Mathematics 3A03 — Real Analysis I

TERM TEST 2 — 26 March 2026

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: _____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of **16 pages** (*i.e.*, **8 double-sided pages**). There are **6 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There are several blank pages at the end; if you use those pages, state which question you are answering.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (Circle the correct answer.) Determine whether each of the following statements is **TRUE** or **FALSE**. Do not justify your answers.

(a) In a metric space, the union of any collection of closed sets is closed.

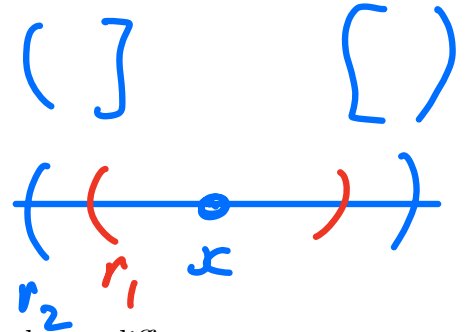
TRUE **FALSE**

(b) If \mathcal{M} is a non-empty set and d is the discrete metric on \mathcal{M} then every set in the metric space (\mathcal{M}, d) is open.

TRUE **FALSE**

(c) The instructor for this course is Mark Carney.

TRUE **FALSE**



(d) In any metric space (\mathcal{M}, d) , if $x \in \mathcal{M}$ and $0 < r_1 < r_2$ then the set difference $B_{r_2}(x) \setminus B_{r_1}(x)$ is neither open nor closed.

TRUE **FALSE**

(e) The series $\sum_{k=1}^{\infty} \frac{x}{k^2}$ converges uniformly on the interval $(-\pi, \pi)$.

TRUE **FALSE**

$$\left| \frac{x}{k^2} \right| \leq \frac{\pi}{k^2} = M_n$$

$$\sum M_n = \pi \sum \frac{1}{k^2} \text{ conv.}$$

(f) If $f_n : [a, b] \rightarrow \mathbb{R}$ is integrable for each n , and $f_n \rightarrow f$ uniformly, then f is integrable and $\int_a^b f_n \rightarrow \int_a^b f$.

TRUE **FALSE**

[6] **QUESTION 2.** Let ℓ^∞ denote the space of all bounded sequences of real numbers, with the sup norm. Thus, if $x = (x_n) \in \ell^\infty$ then

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

The subset E is defined by

$$E = (0, 1)^\infty = \{(x_n) \in \ell^\infty : \underbrace{0 < x_n < 1}_{\text{for all } n \in \mathbb{N}}\}.$$

[3] (a) Give an example of a point x in E° , the interior of E , and prove that $x \in E^\circ$.

Let $x = (\frac{1}{2}, \frac{1}{2}, \dots) \in E \subseteq \ell^\infty$. Let $\varepsilon = \frac{1}{4}$.

Claim: $B_{\frac{1}{4}}(x) \subseteq E$. $\forall n$

Pf: Let $y \in B_{\frac{1}{4}}(x)$. Then $|y_n - x_n| \leq \sup_m |y_m - x_m|$

So, $-\frac{1}{4} < y_n - x_n < \frac{1}{4} \Rightarrow \frac{1}{4} < y_n < \frac{3}{4} \Rightarrow 0 < y_n < 1 \Rightarrow y_n \in E$. $= d(y, x) < \frac{1}{4}$.

[3] (b) Give an example of a point y in ∂E , the boundary of E , and prove that $y \in \partial E$.

Let $y = (0, 0, \dots)$. Let $\varepsilon > 0$.

WTS: ① $B_\varepsilon(y) \cap (\ell^\infty \setminus E) \neq \emptyset$ AND

② $B_\varepsilon(y) \cap \bar{E} \neq \emptyset$.

For ①, $y \in B_\varepsilon(y) \cap (\ell^\infty \setminus E)$.

For ②. Assume $\varepsilon < \frac{1}{2}$.

... Continued...

Then $z = (\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \dots) \in B_\varepsilon(y) \cap E$.

$0 < \frac{\epsilon}{2} < 1$, so $y \in E$. And $|\frac{\epsilon}{2} - d| = \frac{\epsilon}{2} < \epsilon$,
 so $y \in B_\epsilon(y)$.

[10] QUESTION 3.

- [2] (a) State the definitions of **pointwise convergence** and **uniform convergence** for a sequence of functions $f_k : D \rightarrow \mathbb{R}$ converging to a function $f : D \rightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}$.

$f_k \rightarrow f$ pointwise if $\forall x \in D, \lim_{k \rightarrow \infty} f_k(x) = f(x)$.

$f_k \rightarrow f$ uniformly if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N$
 implies $|f_n(x) - f(x)| < \epsilon \quad \forall x \in D$.

- [4] (b) Suppose that for each $k \in \mathbb{N}$, the function $f_k : D \rightarrow \mathbb{R}$ is unbounded on the domain $D \subseteq \mathbb{R}$. If $f_k \rightarrow f$ **pointwise** on D , must f be unbounded on D ? Either prove that f must be unbounded or give a counterexample.

NO. Define $f_k : (0, 1) \rightarrow \mathbb{R}$ by

$$f_k(x) = \frac{1}{kx}. \quad \text{Then } \lim_{x \rightarrow 0^+} f_k(x) = +\infty$$

$\forall k$, so each f_k is unbounded. BUT,

$$\lim_{k \rightarrow \infty} f_k(x) = \lim_{k \rightarrow \infty} \frac{1}{kx} = \frac{1}{+\infty} = 0,$$

so $f_k \rightarrow 0$ pointwise ... Continued...

and the constant map 0 is bounded.

- [4] (c) Again suppose that for each $k \in \mathbb{N}$, the function $f_k : D \rightarrow \mathbb{R}$ is unbounded on the domain $D \subseteq \mathbb{R}$. If $f_k \rightarrow f$ **uniformly** on D , must f be unbounded on D ? Either prove that f must be unbounded or give a counterexample.

Yes. Assume for contradiction f is bounded by $M > 0$. Let $N \in \mathbb{N}$ be st. $\sup_x |f_k(x) - f(x)| < 1 \quad \forall k \geq N$. We know f_N is unbounded, so $\exists x_0 \in D$ s.t. $|f_N(x_0)| > M+1$. But

$$M+1 < |f_N(x_0)| = |f_N(x_0) - f(x_0) + f(x_0)|$$

$$\leq |f_N(x_0) - f(x_0)| + |f(x_0)|$$

$$< 1 + M \neq M+1 < M+M.$$

... Continued...

CONTRADICTION!

[9] QUESTION 4. For each $k \in \mathbb{N}$, define $f_k : (0, \infty) \rightarrow \mathbb{R}$ via

$$f_k(x) = 2^k \sin\left(\frac{1}{3^k x}\right).$$

[5] (a) Prove that the series $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a, \infty)$ for any $a > 0$.

Hint: You may use the fact that $|\sin t| \leq |t|$ for all $t \in \mathbb{R}$.

M-Test. Let $x \in (0, \infty)$.

$$\begin{aligned} \text{Then } \forall k, |f_k(x)| &= \left| 2^k \sin\left(\frac{1}{3^k x}\right) \right| \\ &= 2^k \left| \sin\left(\frac{1}{3^k x}\right) \right| \leq 2^k \left| \frac{1}{3^k x} \right| \end{aligned}$$

$$\text{But } x \geq a \Rightarrow \frac{1}{x} \leq \frac{1}{a} \Rightarrow \frac{1}{|x|} \leq \frac{1}{a}.$$

$$\text{So, } |f_k(x)| \leq 2^k \frac{1}{3^k} \cdot \frac{1}{a}.$$

$$\text{But, } \sum 2^k \frac{1}{3^k} \cdot \frac{1}{a} = \frac{1}{a} \sum \left(\frac{2}{3}\right)^k$$

... Continued

$= \frac{1}{a} \cdot \frac{3}{3-2} = \frac{3}{a}$ geometric series

So, $\sum 2^k \sin\left(\frac{1}{3^k x}\right)$ converges
by M M M M M test!

[4] (b) Prove that the series $\sum_{k=1}^{\infty} f_k(x)$ does not converge uniformly on $(0, \infty)$.

Hint: Use the facts that

(1) $\sin\left(\frac{\pi}{2}\right) = 1$;

(2) if $\sum_{k=1}^{\infty} f_k$ converges uniformly then $f_k \rightarrow 0$ uniformly as $k \rightarrow \infty$.

Know: $f_k \not\rightarrow 0$ uniformly if

$\exists \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$, $\exists k \geq n$ and $\exists x_0 \in (0, \infty)$
s.t. $|f_k(x_0)| \geq \varepsilon$.

Let $\varepsilon = 1$. Let $N \in \mathbb{N}$.

Let $k = N$. Let $x_0 = \frac{2}{3^k}$.

$$f_k(x) = 2^k \sin\left(\frac{1}{3^k x}\right)$$

Then, $|f_k(x_0)| = |2^k (1)| \geq 2^0 = 1$.

So $f_k \not\rightarrow 0$ uniformly ... Continued ...

$\Rightarrow \sum f_k(x)$ does not converge unif. on $(0, \infty)$.

[10] QUESTION 5.

[2] (a) State the formal definition of "a metric space (\mathcal{M}, d) ", where \mathcal{M} is a non-empty set and $d: \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$.

(b) Let $\widehat{\mathbb{R}}$ denote the set of real numbers \mathbb{R} together with two extra "points at infinity". Thus, this set of **extended real numbers** is

$$\widehat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

Now define a function $h: \widehat{\mathbb{R}} \rightarrow [-1, 1]$ by

$$h(x) = \frac{x}{1+|x|} \text{ for } x \in \mathbb{R}, \quad h(-\infty) = -1, \quad h(+\infty) = 1.$$

[3] Prove that h is one-to-one on $\widehat{\mathbb{R}}$.

If $x \leq 0$, then $h(x) = \frac{x}{1-x}$.

$$\Rightarrow h(x) = \frac{1}{1-x} - \frac{x}{(1-x)^2} > 0$$

$\underbrace{\hspace{1.5cm}}_{>0} \quad \underbrace{\hspace{1.5cm}}_{>0}$

$\Rightarrow h$ is strictly increasing on $(-\infty, 0]$ Continued...

Similar: h is strict. inc. on $[0, \infty)$.

\Rightarrow h strict. inc. on $(-\infty, \infty)$.

Also, $|h(x)| = \frac{|x|}{1+|x|} < 1 \Rightarrow h(x) \neq 1, -1 \forall x \in (-\infty, \infty)$

But $h(-\infty) = -1, h(\infty) = 1$. CONCLUSION: h strict. inc.
(c) Given the function h defined in part (b), define a function $d: \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} \rightarrow \mathbb{R}$ by \Rightarrow c-1.

$$d(x, y) = |h(x) - h(y)|.$$

[1] Compute $d(-1, 1)$ and $d(-\infty, +\infty)$.

$$d(-1, 1) = |h(-1) - h(1)| = \left| \frac{-1}{1+1} - \frac{1}{1+1} \right| = 1.$$

$$d(-\infty, \infty) = |h(-\infty) - h(\infty)| = |-1 - 1| = 2.$$

[4] (d) Prove that $(\widehat{\mathbb{R}}, d)$ is a metric space.

① $d(x, y) = |\text{something}| \geq 0$.

② $d(x, y) = |h(x) - h(y)| \leq |h(x) - h(z)|$

③ $d(x, x)$
 $= |h(x) - h(x)| = 0$.
 $+ |h(z) - h(x)|$
 $= d(x, z) + d(z, x)$

④ $d(x, y) = 0 \Rightarrow |h(x) - h(y)| = 0 \Rightarrow h(x) = h(y)$
 $\Rightarrow x = y$ because h 1-1.

⑤ $d(x, y) = |h(x) - h(y)| = |h(y) - h(x)| = d(y, x)$... Continued ...

[9] **QUESTION 6.** Let X be a vector space on which a norm $\|\cdot\|$ is defined. Let d be the metric induced by this norm, *i.e.*,

$$d(x, y) = \|x - y\|.$$

[4] (a) Prove that the function $f : X \rightarrow \mathbb{R}$ defined by

$$f(x) = \|x\|$$

is continuous on the metric space (X, d) .

... *Continued...*

- [5] (b) Prove that a norm can fail to be continuous with respect to a metric other than the metric induced by the norm.

Hint: Consider $X = \mathbb{R}$ with the norm $\|x\| = |x|$. Define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(n) = \frac{1}{n} \text{ for } n \in \mathbb{N}, \quad \phi(x) = x \text{ otherwise.}$$

Now define $\rho : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\rho(x, y) = |\phi(x) - \phi(y)|.$$

You may assume that ρ is a metric.

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