

Question 1: Let

$$C[0,1] = \left\{ \begin{array}{l} \text{continuous} \\ \text{maps} \\ [0,1] \rightarrow \mathbb{R} \end{array} \right\},$$

with the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)| dx \quad \forall f,g \in C[0,1].$$

You may assume d is a metric. Let

$$E = \{f \in C[0,1] \mid 0 < f(x) < 1 \quad \forall x \in [0,1]\} \\ \subseteq C[0,1].$$

Last tutorial, we showed that $E^\circ = \emptyset$.

Show that $\overline{E} = \partial E$.

Solution: Recall that $\bar{E} = E^\circ \cup \partial E$.

But $E^\circ = \emptyset$. So, $\bar{E} = \emptyset \cup \partial E = \partial E$.

Question 2: Show that $E \subseteq \partial E$.

Solution: Recall that $E \subseteq \bar{E}$. But $\bar{E} = \partial E$.

So $E \subseteq \partial E$.

Question 3^{*}: Is it true that $\partial E \subseteq E$?

Solution: Let $f \in \partial E$. Then $\partial E = \bar{E}$, so $f \in \bar{E}$.

i.e., \exists a sequence of points in E , call it

$f_1, f_2, f_3, \dots \in E$, such that $f_n \xrightarrow{d} f$.

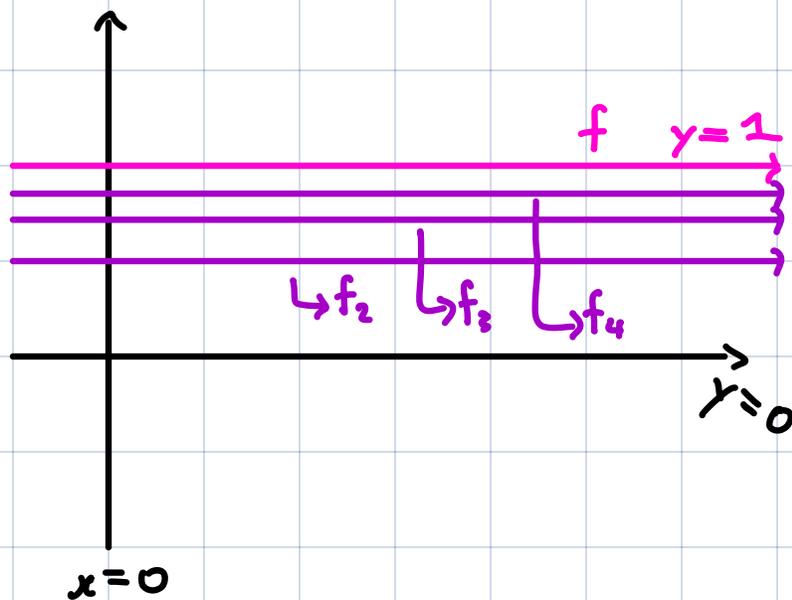
Let $f \in C[0, 1]$ be the map $f(x) = 1 \forall x \in [0, 1]$.

For $n = 2, 3, 4, \dots$ let f_n be the map $f_n(x) = 1 - \frac{1}{n}$.

Note: $f_n \in E \quad \forall n$.

Note: $f \notin E$.

WTS: $f_n \xrightarrow{d} f$.



$$\text{But } d(f_n, f) = \int_0^1 |f_n(x) - f(x)| dx = \frac{1}{n}$$

$\forall n$, so indeed $f_n \xrightarrow{d} f$.

f is a point in \bar{E} but not in E . Since

$\bar{E} = \partial E$, we conclude that $\partial E \not\subset E$

(since $f \in \partial E$ but $f \notin E$).

Question 4: Is E open?

Solution: No! Since E open iff $E = E^\circ$, but

$E^\circ = \emptyset$, while $E \neq \emptyset$.

Question 5: Is E closed?

Solution: No! Since E closed iff $E = \bar{E}$. But

f from Question 3 $\ni (f(x) = 1 \ \forall x \in [0, 1])$

has $f \in \bar{E}$ and $f \notin E$. So, $\bar{E} \neq E$.

Question 6*: Let $l_\infty = \left. \begin{array}{l} \text{bounded} \\ \text{sequences} \\ \text{of real numbers} \end{array} \right\}$

with the metric
$$x = (x_0, x_1, x_2, \dots)$$

$d(x, y) = \sup_{n \in \{0, 1, 2, \dots\}} |x_n - y_n| \ \forall x, y \in l_\infty$. You may

assume d is a metric. Let

$E = \{ \text{Cauchy sequences} \} \subseteq l_\infty$.

Is E° empty?

Solution: Yes. Let $x \in E$. For $n \in \mathbb{N}$, let

$x_n = \left(\frac{1}{n}, -\frac{1}{n}, \frac{1}{n}, -\frac{1}{n}, \dots\right)$. Then $x + x_n \xrightarrow{d} x$

but $x + x_n \notin E \ \forall n$. So, $x \notin E^\circ$. We conclude

that $E^\circ = \emptyset$.

Question 7: Is E open?

Solution: No! E open iff $E = E^\circ$. But $E^\circ = \emptyset$,

while $E \neq \emptyset$.

Question 8:^{**} Is E closed?

Solution: Yes! Assume for contradiction not.

Then \exists a sequence $x^n \in E$ converging

to some $x \in \ell_\infty \setminus E$. i.e., x is a bounded

but non-Cauchy sequence, and $x^n \xrightarrow[n \rightarrow \infty]{d} x$.

But Cauchy \Leftrightarrow Convergence. So, x is bounded

and non-convergent. Since x is bounded, $S = \limsup(x)$

and $I = \liminf(x)$ are well-defined real numbers.

Since x is non-convergent, $S \neq I$. But it is a

general fact that $S \geq I$. So, $S > I$ ($\Leftrightarrow S - I > 0$).

Let x_{m_k} and x_{l_k} be subsequences of x converging

to S and I respectively. Take $K \in \mathbb{N}$ large

such that $k \geq K$ implies $|x_{m_k} - S| < \frac{S - I}{8}$

and $|x_{l_k} - I| < \frac{S - I}{8}$. Take $N \in \mathbb{N}$ large

such that $n \geq N$ implies $d(x^n, x) < \frac{S - I}{8}$

(can do this because $x^n \rightarrow x$). Let

$$k \geq \max\{K, N\}.$$

$$\text{Then } |S - I| \leq |S - x_{m_k}| + |x_{m_k} - x_{l_k}| + |x_{l_k} - I|$$

$$< \frac{S-I}{8} + |x_{m_k} - x_{l_k}| + \frac{S-I}{8}$$

$$= \frac{S-I}{4} + |x_{m_k} - x_{l_k}|$$

$$\leq \frac{S-I}{4} + |x_{m_k} - x_{m_k}^k| + |x_{m_k}^k - x_{l_k}^k| + |x_{l_k}^k - x_{l_k}|$$

$$< \frac{S-I}{4} + \frac{S-I}{8} + |x_{m_k}^k - x_{l_k}^k| + \frac{S-I}{8}$$

$$= \frac{S-I}{2} + |x_{m_k}^k - x_{l_k}^k|$$

$$\Rightarrow 0 < \frac{S-I}{2} < |x_{m_k}^k - x_{l_k}^k| \quad \forall k \geq \max\{K, N\}$$

$\Rightarrow x^k$ is not Cauchy for all $k \geq \max\{K, N\}$,

contradicting our assumption that $x^k \in E$.

Question 9: Is $l_\infty \setminus E$ open?

Solution: Yes! For l_2 is closed.

Question 10^{*}: Let $l_2 \subseteq l_\infty$ be the subset of l_∞ defined by

$$l_2 = \left\{ \begin{array}{l} \text{sequences } x_0, x_1, x_2, \dots \\ \text{of real numbers such} \\ \text{that } \sum_{n=0}^{\infty} x_n^2 \text{ converges} \end{array} \right\}.$$

Still using the supremum metric d , is l_2 open?

Solution: No! $0 = (0, 0, 0, \dots) \in l_2$. But

$$y^n = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots \right) \xrightarrow[n \rightarrow \infty]{d} 0, \text{ while}$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{n} \right)^2 = \infty \Rightarrow y^n \notin l_2 \quad \forall n.$$

Question 11^{*}: Is l_2 closed?

Solution: No! Let $y^n = (\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots, \frac{1}{\sqrt{n}}, 0, 0, \dots)$.

Then $y^n \in \ell^2$ for all n . But $y^n \xrightarrow[n \rightarrow \infty]{d} (\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \dots) \notin \ell^2$.

Question 12: * We can also equip ℓ_2 with the

metric $d'(x, y) = \sum_{k=0}^{\infty} (x_k - y_k)^2$. Let $y^n \in \ell_2$

be a sequence. Let $y \in \ell_2$. Show that

$y^n \xrightarrow{d'} y$ implies $y^n \xrightarrow{d} y$ but not

conversely.

Solution: Assume $y^n \xrightarrow{d'} y$. Then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} (y_k^n - y_k)^2 = 0. \text{ But}$$

$y^n - y \in \ell_2$, which implies $\|y^n - y\|_{\infty} = |y_k^n - y_k|$ elaborate

for some $k_0 \in \mathbb{N}$. So,

$$\begin{aligned} \sum_{k=0}^{\infty} (\gamma_k^n - \gamma_k)^2 &= \sum_{k \neq k_0} (\gamma_k^n - \gamma_k)^2 \\ &\quad + (\gamma_{k_0}^n - \gamma_{k_0})^2 \\ &= \sum_{k \neq k_0} (\gamma_k^n - \gamma_k)^2 \\ &\quad + \|\gamma^n - \gamma\|_{\infty}^2 \\ &\geq \|\gamma^n - \gamma\|_{\infty}^2 \end{aligned}$$

So, $\lim_{n \rightarrow \infty} \|\gamma^n - \gamma\|_{\infty}^2 = 0$ as well, which

implies $\lim_{n \rightarrow \infty} \|\gamma^n - \gamma\|_{\infty} = 0$. i.e., $\gamma^n \xrightarrow{d} \gamma$,

as desired. Conversely, $(\frac{1}{n}, \frac{1}{n}, \dots) \xrightarrow{d} (0, 0, \dots)$ but $(\frac{1}{n}, \dots) \not\xrightarrow{d'} (0, \dots)$.

Question 13: Let $M[0, 1] = \left\{ \begin{array}{l} \text{bounded maps} \\ [0, 1] \rightarrow \mathbb{R} \end{array} \right\}$.

Note: $C[0, 1] \subseteq M[0, 1]$ by the EVT

(where $C[0,1] = \left\{ \begin{array}{l} \text{continuous maps} \\ [0,1] \rightarrow \mathbb{R} \end{array} \right\}$).

Equip $M[0,1]$ with the sup metric

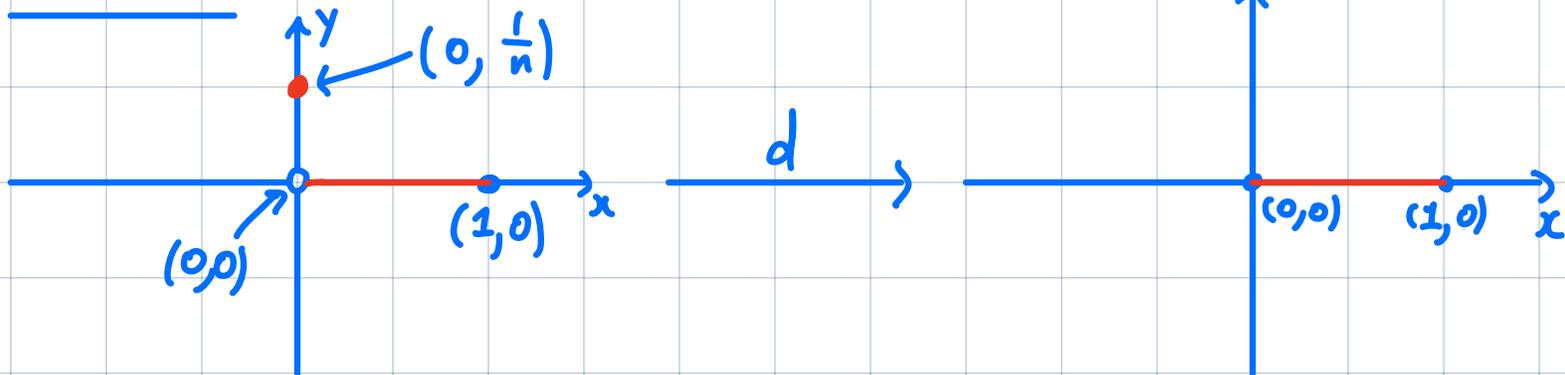
$$d(f,g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)| \quad \forall f,g \in M[0,1].$$

Is $C[0,1]$ a closed subset of $M[0,1]$?

Solution: Yes! Convergence in $M[0,1]$ corresponds to uniform convergence. The uniform limit of continuous maps is again continuous.

Question 14:* Is $C[0,1]$ open?

Solution: No!



$$f_n(x) = \begin{cases} \frac{1}{n}, & x=0 \\ 0, & 0 < x \leq 1 \end{cases} \text{ is in } M[0,1] \text{ } \forall n \text{ and}$$

$f_n \xrightarrow{d} \text{zero map}$, but zero map is in $C[0,1]$.

Question 15*: In Question 9, we saw that

$\mathcal{l}_\infty \setminus E$ is an open subset of \mathcal{l}_∞ , where

$$E = \{\text{Cauchy sequences}\}, \mathcal{l}_\infty = \{\text{bounded sequences}\},$$

and \mathcal{l}_∞ is topologized by the metric

$$d(x, y) = \sup_{n \in \mathbb{N}} \{|x_n - y_n|\} \quad \forall x, y \in \mathcal{l}_\infty.$$

Find an explicit point $x \in \mathcal{l}_\infty \setminus E$, as well

as an explicit real number $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq \mathcal{l}_\infty \setminus E.$$

Solution: Let $x = (1, -1, 1, -1, \dots)$. Let $\varepsilon = \frac{1}{2}$.

Assume $y \in B_\varepsilon(x)$. Then

$$2 = |1 - (-1)| \leq \underbrace{|1 - x_{2k+1}|}_{< \frac{1}{2}} + |x_{2k+1} - x_{2k}| + \underbrace{|x_{2k} - (-1)|}_{< \frac{1}{2}}$$

$$\Rightarrow 1 \leq |x_{2k+1} - x_{2k}|$$

$\Rightarrow y$ is not Cauchy. Hence, $y \in \mathcal{L} \setminus E$.

Question 16: TRUE OR FALSE:

If (X, d) is a metric space, then $U \subseteq X$ is open if and only if $U^\circ = U$.

Question 17: TRUE OR FALSE :

Let $U \subseteq X$ be open and let $x \in U$.

Then for every $\varepsilon > 0$, there is some $y \in X \setminus U$

such that $y \in B_\varepsilon(x)$.

Question 18: Let $x_n \in X$ be a sequence of

points converging to some $x \in E \subseteq X$. Assume

$x_n \in X \setminus E$ for all x_n . We can conclude that

$x \dots$

BELONGS
TO THE
INTERIOR
OF E .

DOES NOT
BELONG
TO THE
INTERIOR
OF E .

Question 19: TRUE OR FALSE :

If (X, d) is a metric space, then $U \subseteq X$ is open if and only if $U \cap \partial U = \emptyset$.

Question 20: A subset $U \subseteq X$ is open if and only if $X \setminus U$ is... closed!

Question 21: A subset $U \subseteq X$ is closed if and only if $X \setminus U$ is... open!

Question 22: TRUE OR FALSE :

Jeff is a great TA.

Question 23: *

9.3.22 A sequence of functions $\{f_n\}$ is said to be *uniformly bounded* on an interval $[a, b]$ if there is a number M so that

$$|f_n(x)| \leq M$$

for every n and also for every $x \in [a, b]$. Show that a uniformly convergent sequence $\{f_n\}$ of continuous functions on $[a, b]$ must be uniformly bounded. Show that the same statement would not be true for pointwise convergence.

Solution: Assume $\{f_n\}$ is a uniformly convergent sequence of continuous functions.

Then $f =$ uniform limit of f_n exists and is continuous. So, the EVT asserts that $\exists M \in \mathbb{R}^+$

s.t. $|f(x)| < M \quad \forall x \in [a, b]$.

$$\begin{aligned} \text{So, } |f_n(x)| &< |f_n(x) - f(x)| + |f(x)| \\ &< |f_n(x) - f(x)| + M. \end{aligned}$$

Let N_0 be s.t. $n \geq N_0$ implies

$$\sup_{a \leq x \leq b} \{|f_n(x) - f(x)|\} < 1.$$

$$\text{Then } |f_n(x)| < |f_n(x) - f(x)| + M$$

$$< \underline{1} + M \quad \forall n \geq N_0$$

$$\forall n, \text{ let } M_n \in \mathbb{R}^+ \text{ be s.t. } |f_n(x)| < M_n \\ \forall x \in [a, b].$$

(exists by EVT).

Then $\forall n$ and x , we have

$$|f_n(x)| \leq \max \{ M+1, M_0, M_1, M_2, \dots, \\ M_{N-1} \},$$

proving the uniform boundedness condition.

To see that the statement does not hold for pointwise convergence, let $[a, b] = [0, 1]$ with

$$f_n(x) = \min \left\{ nx, \frac{1}{x} \right\}.$$

Question 24: A union of open sets is... **open!**

Question 25:*

9.3.18 Verify that the series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$$

converges uniformly on all of \mathbb{R} .

Solution: M-test!

$$\left| \frac{\cos(kx)}{k^2} \right| \leq \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges as a p-series.

So, the series converges uniformly.

Question 25: Let $p \in \mathbb{R}$. The series

$$\sum_{k=0}^{\infty} \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

converges if and only if... **$p > 1$** .

Question 27: TRUE OR FALSE

If $f_n \rightarrow f$ pointwise, then $f_n \rightarrow f$ uniformly.

Question 28: TRUE OR FALSE:

If $f_n \rightarrow f$ uniformly, then $f_n \rightarrow f$ pointwise.

Question 29: TRUE OR FALSE:

If $f_n \rightarrow f$ pointwise and $f_n \rightarrow g$ uniformly, then $f=g$.

Question 30*:

9.3.1 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = \frac{x^n}{1+x^n}.$$

Solution: First look at pointwise convergence (easier).

$$|x| < 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} \stackrel{\text{direct substitution}}{=} \frac{0}{1+0} = 0.$$

$$x = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = \frac{1}{1+1} = \frac{1}{2}.$$

$x = -1 \Rightarrow \frac{x^n}{1+x^n}$ not even defined for n

odd. So $\lim_{n \rightarrow \infty} \frac{x^n}{1+x^n}$ DNE!

$$|x| > 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = 1.$$

Now look at uniform convergence:

BEST CASE SCENARIO:

$$\frac{x^n}{1+x^n} \xrightarrow{\text{uniformly}} 0 \text{ on } (-1, 1) \quad \textcircled{1} \text{ and}$$

$$\frac{x^n}{1+x^n} \xrightarrow{\text{uniformly}} 1 \text{ on } \{x \in \mathbb{R} \mid |x| > 1\}. \quad \textcircled{2}$$

For $\textcircled{1}$, let $\varepsilon > 0$ and let $0 < R < 1$. Let $|x| < R$.

$$\begin{aligned} \text{Then } \left| \frac{x^n}{1+x^n} \right| &= \frac{|x|^n}{|1+x^n|} < \frac{R^n}{|1+x^n|} \\ &\leq \frac{R^n}{1-|x|^n} < \frac{R^n}{1-R^n} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Choose N large such that $n \geq N$

$$\Rightarrow \frac{R^n}{1-R^n} < \varepsilon. \text{ Then } n \geq N \text{ implies}$$
$$\left| \frac{x^n}{1+x^n} - 0 \right| < \varepsilon \quad \forall |x| < R.$$

CONCLUSION: $\forall 0 < R < 1$,

we have $\frac{x^n}{1+x^n} \xrightarrow{\text{uniformly}} 0$ on $[-R, R]$.

For (2), let $\varepsilon > 0$. Let $R > 0$ and $|x| > |R+1|$.

$$\begin{aligned} \text{Then } \left| \frac{x^n}{1+x^n} - 1 \right| &= \frac{1}{|1+x^n|} \leq \frac{1}{|x|^n - 1} \\ &< \frac{1}{|1+R|^n - 1} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

So choose N large s.t. $n \geq N$ implies

$$\frac{1}{|1+R|^n - 1} < \varepsilon. \text{ Then } n \geq N \text{ implies}$$

$$\left| \frac{x^n}{x^n + 1} - 1 \right| < \varepsilon \quad \forall |x| > |R+1|.$$

CONCLUSION: $\forall R > 0, \frac{x^n}{x^{n+1}} \xrightarrow{\text{uniformly}} 1$

on $\{x \in \mathbb{R} \mid |x| > 1 + R\}$.

Question 31*:

9.3.2 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = x^2 e^{-nx}.$$

On what sets can you determine uniform convergence? On what sets can you determine uniform convergence for the sequence of functions $n^2 f_n(x)$?

Solution: Start with pointwise!

$$\begin{aligned} x \geq 0 &\Rightarrow 0 \leq \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^2 e^{-nx} \\ &= x^2 \lim_{n \rightarrow \infty} e^{-nx} \\ &= (x^2)(0) \\ &= 0. \end{aligned}$$

So, $x^2 e^{-nx} \xrightarrow{\text{point}} 0$.

$$\begin{aligned}x < \infty &\Rightarrow \lim_{n \rightarrow \infty} f_n(x) = x^2 \lim_{n \rightarrow \infty} e^{-nx} \\ &= x^2 (+\infty) \\ &= +\infty\end{aligned}$$

\Rightarrow NO CONVERGENCE.

Now uniform convergence.

BEST CASE SCENARIO:

$f_n \rightarrow 0$ uniformly on $[0, \infty)$.

Let $\epsilon > 0$. Let $0 < a < b$ and let $x \in [a, b]$.

$$\text{THEN } |x^2 e^{-nx} - 0| = x^2 e^{-nx}$$

$$< b^2 e^{-na} \xrightarrow{n \rightarrow \infty} 0.$$

So choose N large s.t. $n \geq N$ implies

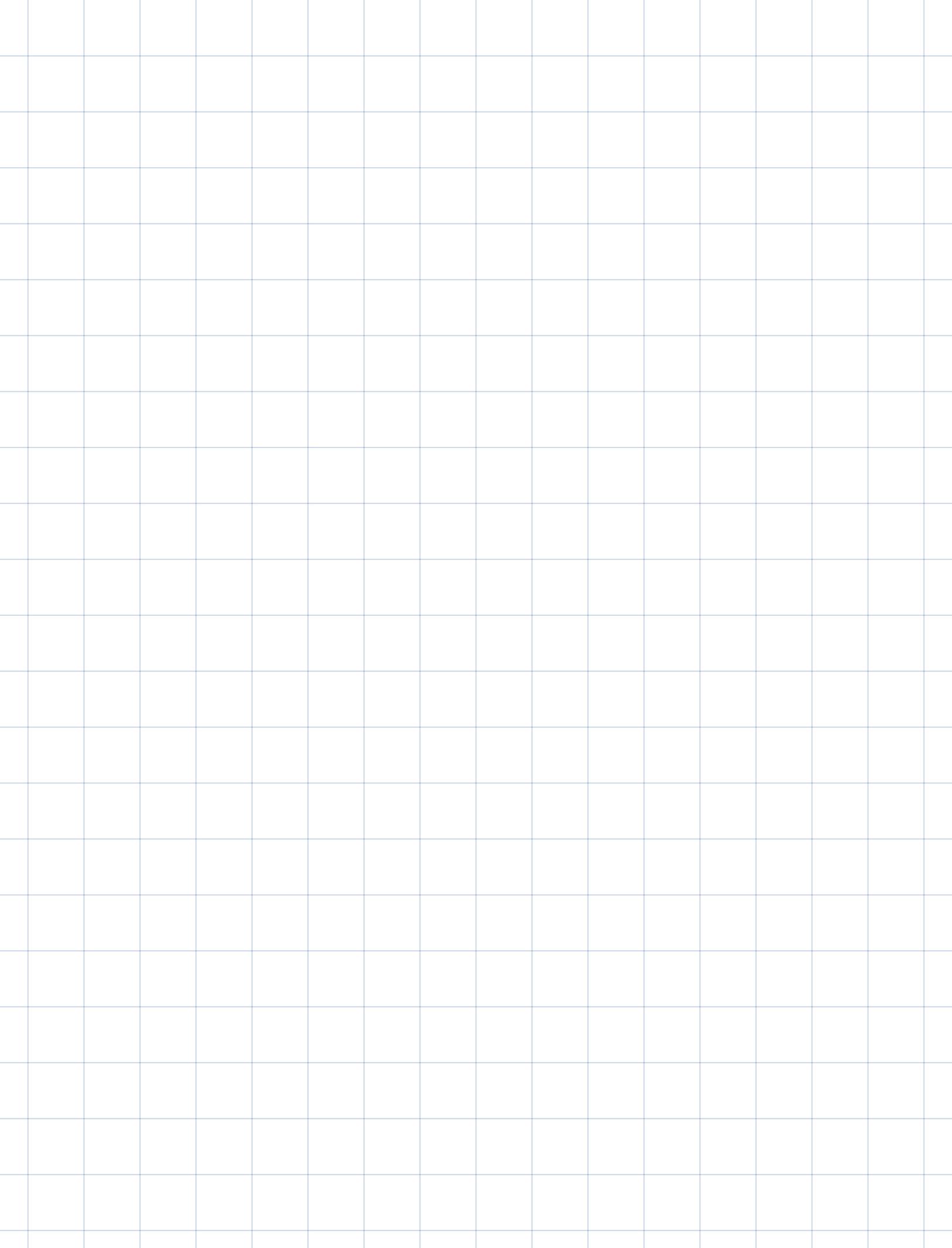
$b^2 e^{-na} < \varepsilon$. Then $n \geq N$ implies

$$|x^2 e^{-nx} - 0| < \varepsilon \quad \forall x \in [a, b].$$

CONCLUSION: $\forall \varepsilon > 0$, $a < b$,

$$x^2 e^{-nx} \xrightarrow{\text{unif.}} 0 \quad \text{on } [a, b].$$

Next, $n^2 f_n(x)$... SOLUTION
INCOMPLETE



Question 32*: Show that the geometric

series $\sum_{n=0}^{\infty} x^n$ does not converge

uniformly on $(-1, 1)$.

Solution: In general, the uniform convergence

of $\sum_{n=0}^{\infty} f_n(x)$ implies $f_n \xrightarrow{\text{unif.}} 0$.

So it suffices to show that

$x^n \not\xrightarrow{\text{unif.}} 0$ on $(-1, 1)$.

Let $\varepsilon = \frac{1}{2}$ and let $N \in \mathbb{N}$. Note that

$\lim_{x \rightarrow 1} x^N = 1$. So, for some $x_0 \in (0, 1)$,

we have $|x_0^N - 1| < \frac{1}{2}$.

So,

$$|x_0^N - 0| = x_0^N$$

$$> \frac{1}{2} = \epsilon.$$

So, $x^n \xrightarrow{\text{unif}} 0$ on $(-1, 1)$,

which implies $\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $(-1, 1)$.

Question 33 *

13.2.1 Which of the following functions defined for pairs of numbers x and y are metrics on \mathbb{R} ?

(a) $d(x, y) = |x| + |y|$

(b) $d(x, y) = (x - y)^2$

Thomson*Bruckner*Bruckner

Elementary Real Analysis, 2nd E

ClassicalRealAnalysis.com

Section 13.2. Metric Spaces—Specific Examples

(c) $d(x, y) = \sqrt{|x - y|}$

(d) $d(x, y) = \min\{1, |x - y|\}$

(e) $d(x, y) = \frac{|x - y|}{1 + |x - y|}$

(f) $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$

SEE NOTE 290

Solution: Sorry... no time to write up a solution!

Question 34 *: Prove or find a counter-

example: If $f_n \xrightarrow{\text{unif.}} f$ and $g_n \rightarrow g$,

then the product functions

$f_n(x)g_n(x)$ converge uniformly to
 $f(x)g(x)$.

Solution: This one's all on you!