Question 1: Let

$$C[0,1] = \left\{ \begin{array}{c} \text{continuous maps} \\ [0,1] \to \mathbb{R} \end{array} \right\},$$

with the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \quad \forall f,g \in C[0,1].$$

You may assume d is a metric. Let

$$E = \{ f \in C[0,1] \mid 0 < f(x) < 1 \ \forall x \in [0,1] \}$$

$$\subseteq C[0,1].$$

Last tutorial, we showed that $E^\circ = \emptyset$.

Show that $\overline{E} = \partial E$.

Question 2: Show that $E \subseteq \partial E$.

Question 3*: Is it true that $\partial E \subseteq E$?

Question 4: Is E open?

Question 5: Is E closed?

Question 6: * Let $l_\infty = \left\{ \begin{matrix} \text{bounded} \\ \text{sequences} \\ \text{of real numbers} \end{matrix} \right\}$

$\rotatebox{90}{$\in$}$

$x = (x_0, x_1, x_2, \dots)$

with the metric

$$d(x,y) = \sup_{n \in \{0,1,2,\dots\}} |x_n - y_n| \quad \forall x,y \in l^\infty.$$

You may assume d is a metric. Let

$$E = \{\text{Cauchy sequences}\} \subseteq l_\infty.$$

Is E° empty?

Question 7: Is E open?

Question 8**: Is E closed?

Question 9: Is $\ell_\infty \setminus E$ open?

Question 10*: Let $\ell_2 \subseteq \ell_\infty$ be the subset of ℓ_∞ defined by

$$\ell_2 = \left\{ \begin{array}{l} \text{sequences } x_0, x_1, x_2, \dots \\ \text{of real numbers such} \\ \text{that } \sum_{n=0}^{\infty} x_n^2 \text{ converges} \end{array} \right\}.$$

Still using the supremum metric d, is ℓ_2 open?

Question 11*: Is ℓ_2 closed?

Question 12:* We can also equip ℓ_2 with the metric $d'(x,y) = \sum_{k=0}^{\infty} (x_k - y_k)^2$. Let $y^n \in \ell_2$ be a sequence. Let $y \in \ell_2$. Show that

sup metric

$$y^n \xrightarrow{d'} y \quad \text{implies} \quad y^n \xrightarrow{d} y \quad \text{but not}$$

conversely.

Question 13: Let $M[0,1] = \left\{ \begin{array}{c} \text{bounded maps} \\ [0,1] \longrightarrow \mathbb{R} \end{array} \right\}$.

Note: $C[0,1] \subseteq M[0,1]$ by the EVT (where $C[0,1] = \left\{ \begin{matrix} \text{continuous maps} \\ [0,1] \longrightarrow \mathbb{R} \end{matrix} \right\}$).

Equip $M[0,1]$ with the sup metric

$$d(f,g) = \sup_{0 \le x \le 1} |f(x) - g(x)| \quad \forall f,g \in M[0,1].$$

Is $C[0,1]$ a closed subset of $M[0,1]$?

<u>Question 14:</u>* Is $C[0,1]$ open?

Question 15*: In Question 9, we saw that $l_\infty \setminus E$ is an open subset of l_∞, where $E = \{\text{Cauchy sequences}\}$, $l_\infty = \{\text{bounded sequences}\}$, and l_∞ is topologized by the metric

$$d(x,y) = \sup_{n \in \mathbb{N}} \{|x_n - y_n|\} \quad \forall x,y \in l_\infty .$$

Find an explicit point $x \in l_\infty \setminus E$, as well as an explicit real number $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq l_\infty \setminus E$.

Question 16: TRUE OR FALSE:

If (X,d) is a metric space, then $U \subseteq X$ is open if and only if $U^{\circ} = U$.

Question 17: TRUE OR FALSE:

Let $U \subseteq X$ be open and let $x \in U$.

Then for every $\varepsilon > 0$, there is some $y \in X \setminus U$

such that $y \in B_\varepsilon(x)$.

Question 18: Let $x_n \in X$ be a sequence of

points converging to some $x \in E \subseteq X$. Assume

$x_n \in X \setminus E$ for all x_n. We can conclude that

x...

BELONGS TO THE INTERIOR OF E.

DOES NOT BELONG TO THE INTERIOR OF E.

Question 19: TRUE OR FALSE:

If (X,d) is a metric space, then $U \subseteq X$ is open if and only if $U \cap \partial U = \emptyset$.

Question 20: A subset $U \subseteq X$ is open if and only if $X \setminus U$ is...

Question 21: A subset $U \subseteq X$ is closed if and only if $X \setminus U$ is...

Question 22: TRUE OR FALSE:

Jeff is a great TA.

Question 23*:

9.3.22 A sequence of functions $\{f_n\}$ is said to be *uniformly bounded* on an interval $[a,b]$ if there is a number M so that

$$|f_n(x)| \leq M$$

for every n and also for every $x \in [a,b]$. Show that a uniformly convergent sequence $\{f_n\}$ of continuous functions on $[a,b]$ must be uniformly bounded. Show that the same statement would not be true for pointwise convergence.

Question 24: A union of open sets is...

Question 25*:

9.3.18 Verify that the series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$$

converges uniformly on all of $\mathbb{R}$.

Question 26: Let $p \in \mathbb{R}$. The series

$$\sum_{k=0}^{\infty} \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

converges if and only if...

Question 27: TRUE OR FALSE

If $f_n \to f$ pointwise, then $f_n \to f$ uniformly.

Question 28: TRUE OR FALSE:

If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Question 29: TRUE OR FALSE:

If $f_n \to f$ pointwise and $f_n \to g$ uniformly, then $f = g$.

Question 30*:

9.3.1 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

Question 31*:

9.3.2 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = x^2 e^{-nx}.$$

On what sets can you determine uniform convergence? On what sets can you determine uniform convergence for the sequence of functions $n^2 f_n(x)$?

Question 32*: Show that the geometric series $\sum_{n=0}^{\infty} x^n$ does <u>not</u> converge uniformly on $(-1, 1)$.

Question 33*:

13.2.1 Which of the following functions defined for pairs of numbers x and y are metrics on $\mathbb{R}$?

(a) $d(x,y) = |x| + |y|$

(b) $d(x,y) = (x-y)^2$

(c) $d(x,y) = \sqrt{|x-y|}$

(d) $d(x,y) = \min\{1, |x-y|\}$

(e) $d(x,y) = \frac{|x-y|}{1+|x-y|}$

(f) $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$

SEE NOTE 290

Question 34*: Prove or find a counter-example: If $f_n \xrightarrow{\text{unif.}} f$ and $g_n \to g$, then the product functions

$f_n(x) g_n(x)$ converge uniformly to

$f(x) g(x)$.