Question 1: Let

$$C[0,1] = \left\{ \begin{matrix} \text{continuous maps} \\ [0,1] \to \mathbb{R} \end{matrix} \right\},$$

with the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \quad \forall f,g \in C[0,1].$$

You may assume d is a metric. Let

$$E = \{ f \in C[0,1] \mid 0 < f(x) < 1 \ \ \forall x \in [0,1] \}$$

$$\subseteq C[0,1].$$

Last tutorial, we showed that $E^\circ = \emptyset$.

Show that $\overline{E} = \partial E$.

In general, $\overline{E} = E^\circ \cup \partial E$.

But $E^\circ = \emptyset$. So, $\overline{E} = \emptyset \cup \partial E$

$\Rightarrow \overline{E} = \partial E$.

Question 2: Show that $E \subseteq \partial E$.

General fact: $E \subseteq \overline{E}$.

But $\overline{E} = \partial E$, So $E \subseteq \partial E$.

Question 3*: Is it true that $\partial E \subseteq E$?

No! Let $f \in C[0,1]$ be defined by

$f(x) = 1$. Let $f_n(x) = 1 - \frac{1}{n}$.

$f_n \in E = \{g \in C[0,1] \,/\, 0 < g(x) < 1 \ \forall x\}$.

Check: $f_n \xrightarrow{d} f$.

So, $f \in \partial E$. But clearly, $f \notin E$.

Question 4: Is E open?

No! E is open iff $E = E^\circ$. But $E^\circ = \emptyset$, while $E \neq \emptyset$.

<u>Question 5</u>: Is E closed?

No! We saw question 3 that $f \in \partial E$, but $f \notin E$. So, E is not closed.

<u>Question 6</u>*: Let $l_\infty = \left\{ \begin{array}{l} \text{bounded} \\ \text{sequences} \\ \text{of real numbers} \end{array} \right\}$ with the metric

$$\begin{array}{c} \cup \\ x = (x_0, x_1, x_2, \dots) \end{array}$$

$d(x,y) = \sup\limits_{n \in \{0,1,2,\dots\}} |x_n - y_n| \quad \forall x, y \in l^\infty$. You may assume d is a metric. Let

$$E = \{\text{Cauchy sequences}\} \subseteq l_\infty.$$

Is E° empty?

Yes! If $x = (x_0, x_1, \dots) \in E$,

then $x_n = (x_0 + \frac{1}{n}, x_1 - \frac{1}{n}, x_2 + \frac{1}{n}, x_3 - \frac{1}{n}, \ldots)$ converges to x (wrt d), but $x_n \notin E$ $\forall n$. So, $x \notin E^\circ$.

<u>Question 7</u>: Is E open?

NO! Open $\Leftrightarrow E = E^\circ$. But $E^\circ = \emptyset$, while $E \neq \emptyset$.

<u>Question 8</u>**: Is E closed?

Yes, E is closed.

$\to$ closed $\quad E \subseteq \ell_\infty$

Question 9: Is $\ell_\infty \setminus E$ open?

Yes! Complement of closed = open.

Question 10*: Let $\ell_2 \subseteq \ell_\infty$ be the subset of ℓ_∞ defined by

$$\ell_2 = \left\{ \begin{array}{l} \text{sequences } x_0, x_1, x_2, \dots \\ \text{of real numbers such} \\ \text{that } \sum_{n=0}^{\infty} x_n^2 \text{ converges} \end{array} \right\}.$$

Still using the supremum metric d, is ℓ_2 open?

NO! Take $0 = (0, 0, 0, \dots) \in \ell_2$.

But $x_n = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots\right) \xrightarrow{d} 0$,

but $x_n \notin \ell_2$ $\forall n$ (check). So, $0 \in \ell_2$ but $0 \notin (\ell_2)^\circ$.

Question 11*: Is ℓ_2 closed?

NO! Consider $x_n = \left(\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{n}}, 0, 0, \dots\right)$.

Then $x_n \xrightarrow{d} \left(\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots\right) \in l_\infty \setminus l_2$.

Since $x_n \in l_2$ $\forall n$ but $\lim x_n \notin l_2$, l_2 is NOT CLOSED.

Question 12:* We can also equip l_2 with the metric $d'(x,y) = \sum_{k=0}^{\infty} (x_k - y_k)^2$. Let $y^n \in l_2$ be a sequence. Let $y \in l_2$. Show that

$$y^n \xrightarrow{d'} y \text{ implies } y^n \xrightarrow{d} y \text{ but not}$$

→ sup metric

conversely. Assume $y^n \in l_2$ s.t. $y^n \xrightarrow{d'} y \in l_2$. WTS: $y^n \xrightarrow{d} y$.

$$\lim_{N \to \infty} \sum_{k=0}^{\infty} (y_k^N - y_k)^2 = 0.$$

But $y^N - y \in l_2$. This implies

$$\sup_{k\in\mathbb{N}}\{|y_k^N - y_k|\} = |y_{k_0}^N - y_{k_0}|$$

for some $k_0 \in \mathbb{N}$. (Elaborate).

So, $$\sum_{k=0}^{\infty}(y_k^N - y_k)^2$$

$$= \sum_{k\neq k_0}(y_k^N - y_k)^2$$

$$+ (y_{k_0}^N - y_{k_0})^2$$

$$\geq (y_{k_0}^N - y_{k_0})^2 = d(y^N, y)^2,$$

so $d(y^N, y) \longrightarrow 0$ by squeeze.

<u>Question 13</u>: Let $M[0,1] = \left\{ \begin{matrix} \text{bounded maps} \\ [0,1] \longrightarrow \mathbb{R} \end{matrix} \right\}$.

Note: $C[0,1] \subseteq M[0,1]$ by the EVT (where $C[0,1] = \left\{ \begin{array}{l} \text{continuous maps} \\ [0,1] \longrightarrow \mathbb{R} \end{array} \right\}$).

Equip $M[0,1]$ with the sup metric

$$d(f,g) = \sup_{0 \le x \le 1} |f(x) - g(x)| \quad \forall f,g \in M[0,1].$$

Is $C[0,1]$ a closed subset of $M[0,1]$?

Yes! Convergence in $M[0,1]$ corresponds to uniform convergence. A uniform limit of continuous functions is continuous

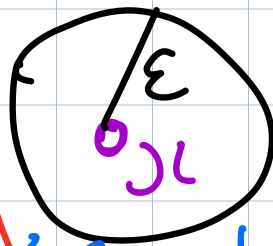
<u>Question 15</u>*: In Question 9, we saw that $l_\infty \setminus E$ is an open subset of l_∞, where $E = \{\text{Cauchy sequences}\}$, $l_\infty = \{\text{bounded sequences}\}$, and l_∞ is topologized by the metric

$$d(x,y) = \sup_{n \in \mathbb{N}} \{|x_n - y_n|\} \quad \forall x,y \in l_\infty .$$

Find an explicit point $x \in l_\infty \setminus E$, as well as an explicit real number $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq l_\infty \setminus E$.

Let $x = (\overset{0}{1}, \overset{1}{-1}, \overset{2}{1}, \overset{3}{-1}, \dots)$.

$\varepsilon = \frac{1}{2}$. Then if



$y \in \ell_\infty$ s.t. $d(x,y) < \varepsilon = \frac{1}{2}$, we have

$$2 = |1-(-1)| \le |1-y_{2k}| + |y_{2k}-(-1)| \quad \ell_\infty$$

$$\le |1-y_{2k}| + |y_{2k}-y_{2k+1}| + |y_{2k+1}-(-1)|$$

$$< \frac{1}{2} + |y_{2k}-y_{2k+1}| + \frac{1}{2}$$

$$\Rightarrow 1 < |y_{2k}-y_{2k+1}| \ \forall k. \text{ So, } y \notin E.$$

Question 16: TRUE OR FALSE :

If (X,d) is a metric space, then $U \subseteq X$ is open if and only if $U^\circ = U$.

Question 17: TRUE OR FALSE :

Let $U \subseteq X$ be open and let $x \in U$.

Then for every $\varepsilon > 0$, there is some $y \in X \setminus U$

such that $y \in B_\varepsilon(x)$.

Question 18: Let $x_n \in X$ be a sequence of

points converging to some $x \in E \subseteq X$. Assume

$x_n \in X \setminus E$ for all x_n. We can conclude that

x...

BELONGS TO THE INTERIOR OF E.

DOES NOT BELONG TO THE INTERIOR OF E.

Question 19: TRUE OR FALSE:

If (X,d) is a metric space, then $U \subseteq X$ is open if and only if $U \cap \partial U = \emptyset$.

Question 20: A subset $U \subseteq X$ is open if and only if $X \setminus U$ is...

Question 21: A subset $U \subseteq X$ is closed if and only if $X \setminus U$ is... closed.

Question 22: TRUE OR FALSE:

Jeff is a great TA.

Question 23*: HIGHLY REC.

9.3.22 A sequence of functions $\{f_n\}$ is said to be *uniformly bounded* on an interval $[a, b]$ if there is a number M so that

$$|f_n(x)| \leq M$$

for every n and also for every $x \in [a, b]$. Show that a uniformly convergent sequence $\{f_n\}$ of continuous functions on $[a, b]$ must be uniformly bounded. Show that the same statement would not be true for pointwise convergence.

Question 24: A union of open sets is... OPEN

Question 25:* HIGHLY REC.

9.3.18 Verify that the series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$$

converges uniformly on all of $\mathbb{R}$.

$\forall x \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$\left| \frac{\cos(kx)}{k^2} \right| = \frac{|\cos(kx)|}{k^2} \leq \frac{1}{k^2}.$$

But $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges as a p-series. SO we get uniform convergence by M-TEST.

Question 26: Let $p \in \mathbb{R}$. The series

$$\sum_{k=0}^{\infty} \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

converges if and only if... $p > 1$

Question 27: TRUE OR FALSE

If $f_n \to f$ pointwise, then $f_n \to f$ uniformly.

Question 28: TRUE OR FALSE:

If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

Question 29: TRUE OR FALSE:

If $f_n \to f$ pointwise and $f_n \to g$ uniformly, then $f = g$.

Question 30*: highly

9.3.1 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = \frac{x^n}{1 + x^n}.$$

Question 31*:

9.3.2 Examine the uniform limiting behavior of the sequence of functions

$$f_n(x) = x^2 e^{-nx}.$$

On what sets can you determine uniform convergence? On what sets can you determine uniform convergence for the sequence of functions $n^2 f_n(x)$?

HIGHLY REC

Question 32*: Show that the geometric series $\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $(-1, 1)$.

Question 33*: HIGHLY

13.2.1 Which of the following functions defined for pairs of numbers x and y are metrics on $\mathbb{R}$?

(a) $d(x, y) = |x| + |y|$

(b) $d(x, y) = (x - y)^2$

(c) $d(x, y) = \sqrt{|x - y|}$

(d) $d(x, y) = \min\{1, |x - y|\}$

(e) $d(x, y) = \frac{|x-y|}{1+|x-y|}$

(f) $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$

SEE NOTE 290

Solution: Sorry... no time to write up a solution!

Question 34*: Prove or find a counter-example: If $f_n \xrightarrow{unif.} f$ and $g_n \to g$, then the product functions

$f_n(x) g_n(x)$ converge uniformly to $f(x) g(x)$.

Solution: This one's all on you!