Tutorial 9

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MIDTERM THURSDAY MARCH 26

Problem 1: Let X be a non-empty set. Does there exist a metric d on X such that every subset of X is open (with respect to the topology induced by d)?

Solution: Yes! Take d to be the discrete metric $d(x,y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$.

Let $U \subseteq X$. Then

$$U = \bigcup_{x \in U} \{x\} = \bigcup_{x \in U} B_1(x)$$

is a union of open balls, so U is open.

Problem 2: Let X be a non-empty set. Does there exist a metric d on X such that, with respect to the topology induced by d, there are only two open sets (namely X itself and $\varnothing$)?

Solution: If $X=\{x\}$ is a singleton, then yes! This follows from Problem 1.

If X contains multiple elements, then no. Indeed, let $x, y \in X$ with $x \neq y$. Then $d(x,y)>0$, so $B_{d(x,y)}(x) \cap B_{d(x,y)}(y)$ is empty! We therefore have at least four open sets; namely

$\emptyset$, $B_{d(x,y)}(x)$, $B_{d(x,y)}(y)$, X.

Problem 3 :

$$C[0,1] = \{\text{continuous maps } [0,1] \to \mathbb{R}\}$$

with the metric $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$.

You may assume d is a metric.

Let $E = \{f \in C[0,1] \mid 0 < f(x) < 1 \ \forall x \in [0,1]\}$.

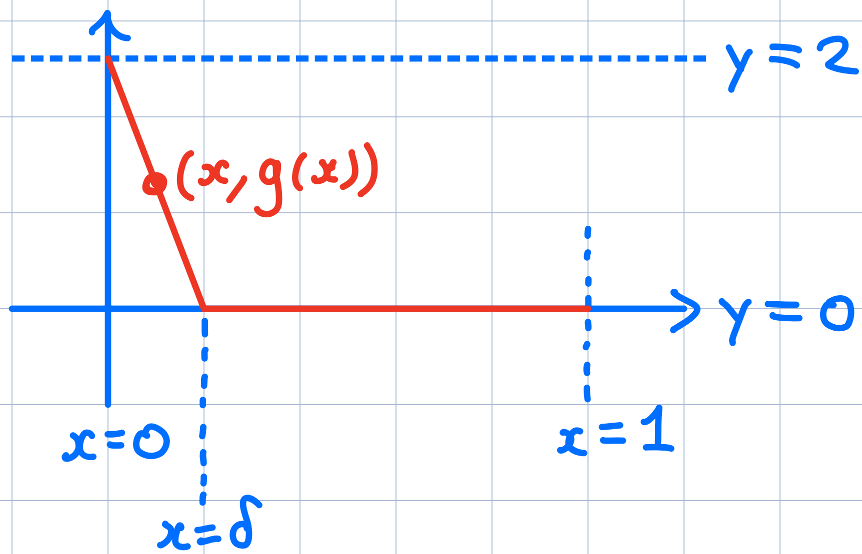
Using the topology induced by d, determine E°.

Try w/ sup-metric
$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$

Solution: $E^\circ = \emptyset$.

Proof: Let $f \in E$, let $\varepsilon > 0$. We want to show that $B_\varepsilon(f) \not\subseteq E$.

Let $g:[0,1]\longrightarrow \mathbb{R}$ be a "bump function" localized at $x=0$.



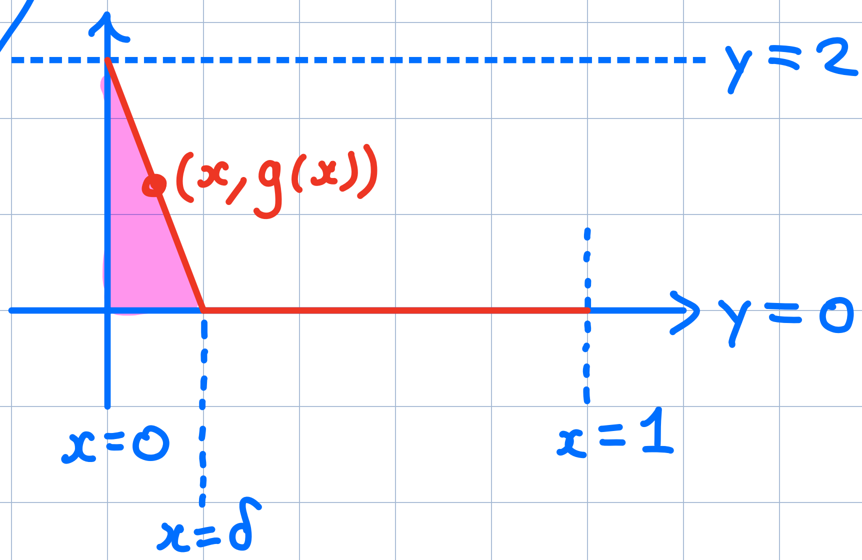
Let $h(x)=f(x)+g(x)$. Then

$$d(f,h)=\int_0^1 |f(x)-h(x)|\,dx$$

$$=\int_0^1 \Big|f(x)-\big(f(x)+g(x)\big)\Big|\,dx$$

$$=\int_0^1 |-g(x)|\,dx$$

$$= \int_0^1 g(x)\,dx$$



$$= \frac{\text{base} \times \text{height}}{2}$$

$$= \frac{\delta \times 2}{2}$$

$$= \delta.$$

Choose $\delta < \varepsilon$. Then $d(f,h) = \delta < \varepsilon$, so $h \in B_\varepsilon(f)$.

Now we want to argue that

$h \notin E = \{ f \in C[0,1] : 0 < f(x) < 1 \ \forall x \}$.

Well $h(0) = f(0) + g(0)$

$> 0 + g(0)$

$= 0 + 2$.

So, $h(0) > 1$, which implies $h \notin E$.

We conclude that $E^\circ = \emptyset$, as desired.

TBB

13.2.2 Let $\mathbb{R}^+$ denote the set of positive real numbers. Show that

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

is a metric on $\mathbb{R}^+$.

<u>Solution</u>: (1) $d(x,x)=0 \quad \forall x \in \mathbb{R}^+$.

<u>Proof</u>: $d(x,x) = \left|\frac{1}{x} - \frac{1}{x}\right| = |0| = 0.$

(2) $d(x,y)=0 \Rightarrow x=y.$

<u>Proof</u>: $d(x,y)=0 \Rightarrow \left|\frac{1}{x} - \frac{1}{y}\right| = 0$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = 0 \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x=y.$$

(3) $d(x,y) = d(y,x)$

<u>Proof</u>: $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{y} - \frac{1}{x}\right|$

$$= d(y,x).$$

(4) $d(x,y) \le d(x,z) + d(z,y) \quad \forall x,y,z \in \mathbb{R}^+$.

Proof: $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y}\right|$

$\le \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| = d(x,z) + d(z,y)$.

TBB

9.5.1 Prove that

$$\lim_{n\to\infty} \int_{\frac{\pi}{2}}^{\pi} \frac{\sin nx}{nx}\, dx = 0.$$

Solution: We use THM 9.29.

Theorem 9.29: *Let $\{f_n\}$ be a sequence of functions Riemann integrable on an interval $[a,b]$. If $f_n \to f$ uniformly on $[a,b]$, then f is Riemann integrable on $[a,b]$ and*

$$\int_a^b f(x)\, dx = \lim_n \int_a^b f_n(x)\, dx.$$

Here $a = \frac{\pi}{2}$, $b = \pi$, $f_n(x) = \frac{\sin(nx)}{nx}$, and

$f(x) = 0$.

Want to show: $f_n \xrightarrow{\text{unif.}} f$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large such that

$\frac{1}{N} < \frac{\pi \varepsilon}{2}$ (Archimedean Property of $\mathbb{R}$).

Let $n \geq N$. Let $x \in [\frac{\pi}{2}, \pi]$. Then

$$|f_n(x) - 0| = |f_n(x)| = \left| \frac{\sin(nx)}{nx} \right|$$

$$\leq \frac{1}{nx} \leq \frac{1}{Nx} \leq \frac{1}{N(\frac{\pi}{2})} \leq \frac{2}{\pi} \cdot \frac{\pi}{2} \cdot \varepsilon = \varepsilon.$$

So, $f_n \xrightarrow{\text{unif.}} 0$, as desired.

Invoking the THM:

$$\lim_{n\to\infty} \int_{\pi/2}^{\pi} \frac{\sin(nx)}{nx}\,dx = \int_{\frac{\pi}{2}}^{\pi} \lim_{n\to\infty} \frac{\sin(nx)}{nx}\,dx$$

$$= \int_{\pi/2}^{\pi} (0)\,dx = 0.$$

TBB

9.5.2 Prove that
$$\int_0^\pi \sum_{n=1}^\infty \frac{\sin nx}{n^2}\,dx = \sum_{n=1}^\infty \frac{2}{(2n-1)^3}.$$

FAKE solution: $\int_0^\pi \sum_{n=1}^\infty \frac{\sin(nx)}{n^2}\,dx$

why is this even integrable?

$$= \sum_{n=1}^\infty \int_0^\pi \frac{\sin(nx)}{n^2}\,dx$$

justify

$$= \sum_{n=1}^\infty \frac{1}{n^2}\int_0^\pi \sin(nx)\,dx$$

$$= \sum_{n=1}^\infty \frac{1}{n^2}\left(-\frac{\cos(nx)}{n}\Big|_0^\pi\right)$$

$$= \sum_{n=1}^\infty \frac{1}{n^2}\left(\frac{1}{n} - \frac{(-1)^n}{n}\right)$$

$$= \sum_{n=1}^\infty \frac{1}{(2n-1)^2}\cdot\left(\frac{2}{2n-1}\right)$$

$$= \sum_{n=1}^\infty \frac{2}{(2n-1)^3}$$

REAL solution: Use Corollary 9.30.

Corollary 9.30: *If an infinite series of integrable functions $\sum_0^\infty f_k$ converges uniformly to a function f on an interval $[a,b]$, then f is also integrable and*

$$\int_a^b f(x)\,dx = \sum_0^\infty \int_a^b f_k(x)\,dx.$$

Here $a=0$, $b=\pi$, $f_k(x) = \dfrac{\sin(kx)}{k^2}$.

Need to show: $\sum_{k=1}^\infty f_k(x)$ converges unif.

to something... to what? I don't know!

M-test: $x \in [0,\pi]$,

$$|f_k(x)| = \left|\frac{\sin(kx)}{k^2}\right| \le \frac{1}{n^2}, \text{ and}$$

$\sum_{n=1}^\infty \frac{1}{n^2}$ converges. So, $\sum_{n=1}^\infty f_k(x)$ converges

uniformly on $[0,\pi]$. Call the limit function f.

Note: Each f_k is integrable on $[0,\pi]$.

So we can apply the Corollary!

So, $\displaystyle\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\,dx$

$$= \int_0^{\pi} \underbrace{f(x)}\,dx$$

Uniform limit of the series

$$= \sum_{n=1}^{\infty} \int_0^{\pi} f_n(x)\,dx \quad \text{(Corollary)}$$

$$= \sum_n \int_0^{\pi} \frac{\sin(nx)}{n^2}\,dx$$

$$= \sum_n \frac{1}{n^2}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right]$$

$$= \sum_n \frac{2}{(2n-1)^3}.$$

from this point on, all computations from the FAKE solution above carry through.

Problem 2: Let X be a non-empty set. Does there exist a metric d on X such that, with respect to the topology induced by d, there are only two open sets (namely X itself and $\emptyset$)?

Solution: Consider $X = \{x\}$ a singleton.

Try the discrete metric.

$$d(x,x) = 0.$$

$$B_1(x) = \{ y \in X \mid d(x,y) < 1 \}$$
$$= \{x\}.$$

So $\{x\}$ is open.

All subsets of X:

$\{\emptyset, \{x\}\}$

both open.

In this case, yes, we can find d.

Now suppose $x, y \in X$, $x \neq y$.

Then $d(x,y) > 0$. So $B_{d(x,y)}(x) \cap B_{d(x,y)}(y) = \emptyset$.

So, there at <u>least</u> 4 open sets

$\emptyset$, $B_{d(x,y)}(x)$, $B_{d(x,y)}(y)$, X.

In this case, d does <u>not</u> exist.

Conclusion: Yes, if $|X| = 1$, No otherwise.

Maybe something if allow infinite values

... BUT WE DON'T.

Problem 3 : Let

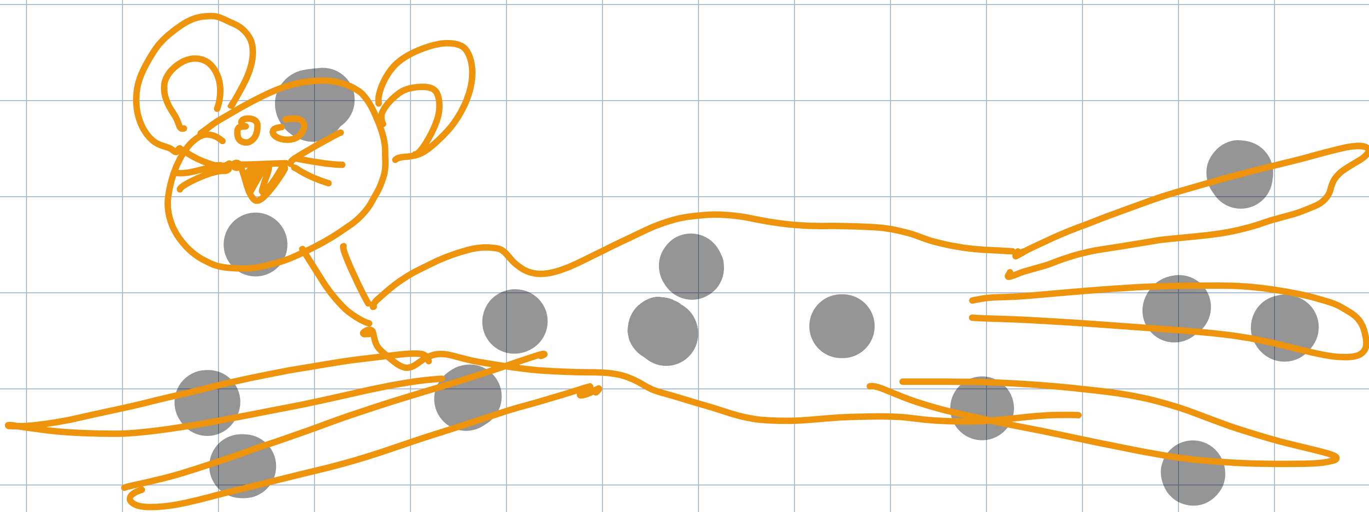
$$C[0,1] = \{\text{continuous maps } [0,1] \to \mathbb{R}\}$$

with the metric $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$.

You may assume d is a metric.

Let $E = \{f \in C[0,1] \mid 0 < f(x) < 1 \ \forall x \in [0,1]\}$.

Using the topology induced by d, determine E°.



Solution: $E = E^\circ$, $\boxed{E^\circ = \emptyset}$, $E^\circ = C[0,1]$

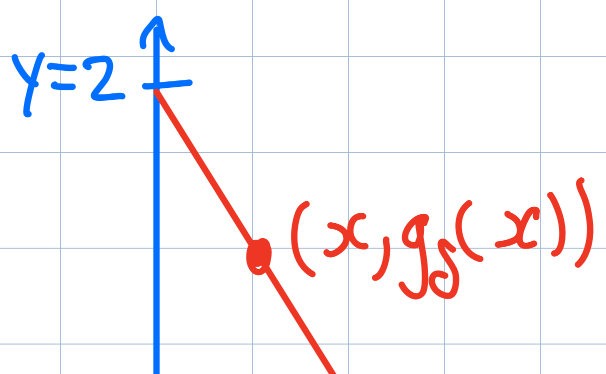
$E^\circ = \{0 \le f(x) \le 1\}$?

Claim: $E^\circ = \emptyset$.

Let $f \in E = \{0 < \lambda(x) < 1\}$ and let $\varepsilon > 0$.

WANT: Some $h \in C[0,1]$ such that $d(f,h) < \varepsilon$ but $h \notin E$.

Let $g_\delta : [0,1] \longrightarrow \mathbb{R}$ be a "bump function" localized at $x=0$.

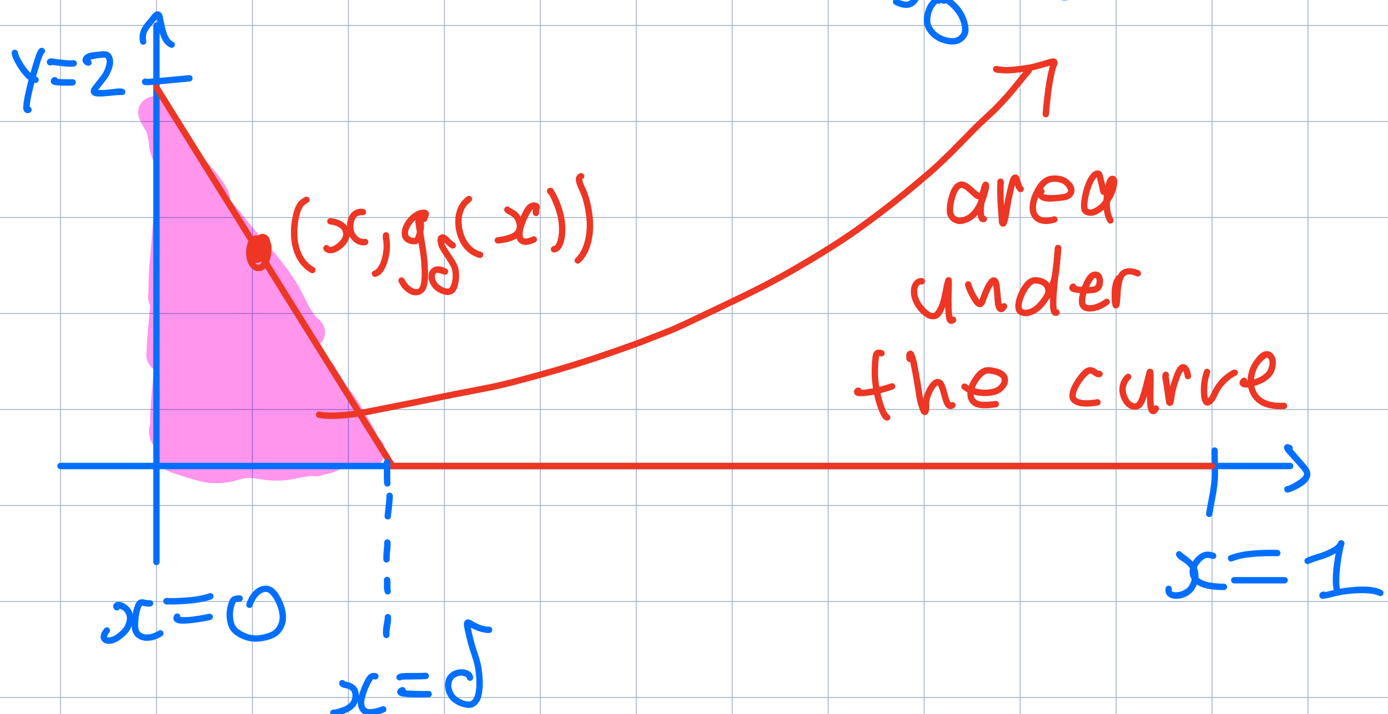


Let $h_\delta(x) = f(x) + g_\delta(x)$.

Then $d(f, h_\delta) = \int_0^1 |f(x) - h_\delta(x)| \, dx$

$$= \int_0^1 |\cancel{f(x)} - (\cancel{f(x)} + g_\delta(x))| \, dx$$

$$= \int_0^1 |-g_\delta(x)| \, dx = \int_0^1 g_\delta(x) \, dx.$$



$$= \frac{\text{base} \times \text{height}}{2} = \frac{\delta \times 2}{2}$$

$$= \delta. \quad < \varepsilon.$$

want

Just set $\delta = \frac{\varepsilon}{100}$.

We proved

$$h_\delta \in B_\varepsilon(f).$$

Last thing to prove: $h_\delta \notin E$.

Indeed $h_\delta(0) = f(0) + g_\delta(0)$

$$> 0 \quad + g_\delta(0)$$

$$= 0 + 2$$

$$= 2 \notin (0, 1).$$

So, $h_\delta \notin E$.

$$\boxed{E^{\circ} = \emptyset.}$$:)

9.5.2 Prove that

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}\, dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}.$$

FAKE SOLUTION:

$$\underbrace{\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\, dx}_{\text{why is this even integrable}} \overset{\text{justify}}{=} \sum_{n=1}^{\infty} \int_0^{\pi} \frac{\sin(nx)}{n^2}\, dx$$

$$= \sum_n \frac{1}{n^2} \int_0^{\pi} \sin(nx)\, dx$$

$$\overset{\text{FTC}}{=} \sum_n \frac{1}{n^2} \left[-\frac{\cos(nx)}{n} \Big|_0^{\pi} \right]$$

$$= \sum_{n} \frac{1}{n^2}\left(\frac{(-1)-(-1)^n}{n}\right)$$

$$= \frac{1}{1^3} + 0 + \frac{1}{3^3} + 0 + \frac{1}{5^3} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}.$$

To make this rigorous, use:

Corollary 9.30: *If an infinite series of integrable functions* $\sum_0^\infty f_k$ *converges uniformly to a function* f *on an interval* $[a,b]$, *then* f *is also integrable and*

$$\int_a^b f(x)\,dx = \sum_0^\infty \int_a^b f_k(x)\,dx.$$

justify

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\,dx = \sum_{n=1}^{\infty} \int_0^{\pi} \frac{\sin(nx)}{n^2}\,dx$$

Argue that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly. (M-test)

This implies (1) That is integrable.

(2) We can interchange $\int_0^\pi \sum = \sum \int_0^\pi$.