# 3A Tutorial 8

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Let $D \subseteq \mathbb{R}$. Let $f: D \longrightarrow \mathbb{R}$, and $\forall n \in \mathbb{N}$

let $f_n : D \longrightarrow \mathbb{R}$.

DEF (pointwise convergence): $f_n$ converges pointwise to $f$ if $\forall x \in D$, $\lim_{n\to\infty} f_n(x) = f(x)$.

DEF (uniform convergence): $f_n$ converges uniformly to $f$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$.

# Assignment 4, Q3

3. Considering $f_n(x) = \dfrac{e^x}{x^n}$, on $(1, \infty)$, which of the following statements are true?

   - ☐ $\{f_n\}$ does not converge;
   - ☑ $\{f_n\}$ converges pointwise;
   - ☐ $\{f_n\}$ converges uniformly;

   If the sequence converges pointwise, determine the pointwise limit on the indicated interval.

Sol'n: Does $\{f_n\}$ convergence pointwise?

Let $x \in (1, \infty)$.

We are interested in the limit

$$\lim_{n\to\infty} \frac{e^x}{x^n}.$$

Claim: $\lim_{n\to\infty} \frac{e^x}{x^n} = 0.$

Pf: Let $\varepsilon > 0$. Then

$$\left|\frac{e^x}{x^n} - 0\right| < \varepsilon$$

$$\Longleftrightarrow$$

$$\frac{e^x}{x^n} < \varepsilon$$

$$\Longleftrightarrow$$

$$x^n > \frac{e^x}{\varepsilon}$$

$$\Longleftrightarrow$$

$$n > \log_x\left(\frac{e^x}{\varepsilon}\right).$$

Choose $N \in \mathbb{N}$ <u>large</u> s.t. $N > \log_x\left(\frac{e^x}{\varepsilon}\right)$

(Archimedean Property of $\mathbb{R}$).

Then certainly for $n \geq N$, we have

$$n > \log_x\left(\frac{e^x}{\varepsilon}\right).$$

So, $\left| \frac{e^x}{x^n} - 0 \right| < \varepsilon$, as desired. $\blacksquare$

Since $x \in (1, \infty)$ was arbitrary, this proves that $\{f_n\} \xrightarrow{\text{pointwise}} 0$.

## Uniform Convergence?

NO!

Claim: $\lim_{x \to \infty} \frac{e^x}{x^n} = +\infty$.

Proof (by induction on $n$):

Base case $n=0$: $\lim_{x \to \infty} \frac{e^x}{x^n} = \lim_{x \to \infty} \frac{e^x}{x^0}$

$$= \lim_{x \to \infty} \frac{e^x}{1} = \lim_{x \to \infty} e^x = +\infty.$$

Inductive step $n \geq 1$:

$$\lim_{n \to \infty} \frac{e^x}{x^n} \overset{L'H}{=\!=} \lim_{n \to \infty} \frac{\frac{d}{dx} e^x}{\frac{d}{dx} x^n}$$

$$= \lim_{n \to \infty} \frac{e^x}{n x^{n-1}}$$

$$= \frac{1}{n} \lim_{n \to \infty} \frac{e^x}{x^{n-1}}$$

$$= \frac{1}{n} (+\infty) \quad \text{(inductive hypothesis)}$$

$$= +\infty. \qquad \blacksquare$$

<u>Conclusion</u>: $\lim_{x \to \infty} \frac{e^x}{x^n} = +\infty.$

<u>Note</u>: $\frac{e^x}{x^n} \xrightarrow{\text{pointwise}} 0$, so if

$$\frac{e^x}{x^n} \xrightarrow{\text{uniformly}} f, \quad \text{then}$$

$$f \equiv 0.$$

So to show $\frac{e^x}{x^n}$ does not converge uniformly on $(1, \infty)$, it suffices to argue that $\frac{e^x}{x^n}$ does not converge uniformly to $f \equiv 0$.

To this end, let $\varepsilon = 1$.

Let $N \in \mathbb{N}$ and let $n = N+1$.

Then $\lim\limits_{x \to \infty} \frac{e^x}{x^n} = +\infty$, so

$\frac{e^y}{y^n} \geq 1 = \varepsilon$ for some $y \in (1, \infty)$.

i.e., $\left|\frac{e^y}{y^n} - 0\right| \geq 1 = \varepsilon$, proving that $\frac{e^x}{x^n}$ does <u>not</u> converge uniformly to $0$

(and therefore that $\frac{e^x}{x^n}$ does not converge uniformly to anything).

Assignment 4, Q4

4. Consider the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}.$$

Which of the following statements are true about this series?

☐ does not converge for any $x \in \mathbb{R}$;

☐ converges pointwise on a non-empty set but not on all of $\mathbb{R}$;

☐ converges pointwise on $\mathbb{R}$;



☐ converges uniformly on a non-empty set but not on all of $\mathbb{R}$;

☐ converges uniformly on $\mathbb{R}$.

# Assignment 4, Q3

3. Considering $f_n(x) = \dfrac{e^x}{x^n}$, on $(1, \infty)$, which of the following statements are true?

- ☐ $\{f_n\}$ does not converge; → false.
- ☐ $\{f_n\}$ converges pointwise; → true, to 0.
- ☐ $\{f_n\}$ converges uniformly; → false.

If the sequence converges pointwise, determine the pointwise limit on the indicated interval.

Sol'n: For pointwise convergence, consider the limit $\lim\limits_{n \to \infty} \dfrac{e^x}{x^n}$.

$$\lim_{n \to \infty} \frac{e^x}{x^n} = e^x \lim_{n \to \infty} \frac{1}{x^n} = 0.$$

$x > 1$

So, $\{f_n\} \xrightarrow{\text{pointwise}} 0$.

## Uniform Convergence?

$$\frac{e^x}{x^n}$$

This converges pointwise to 0.

So <u>if</u> it converges uniformly,

it must converge uniformly to 0.

<u>Claim</u>: $\{f_n\}$ does <u>not</u> converge uniformly to 0.

<u>Pf</u>: Let $\varepsilon = 947$.

Let $N \in \mathbb{N}$ be arbitrary.

Need to find $n \geq N$ such that for some $x \in (1, \infty)$, we have

$$|f_n(x) - 0| \geq \varepsilon = 947.$$

<u>Subclaim</u>: $\forall n$ $\lim_{x \to \infty} \frac{e^x}{x^n} = +\infty$.

Let $n = N$. Then by the Subclaim, there exists $y \in (1, \infty)$ such that

$$\frac{e^y}{y^n} \geq 947.$$

i.e. $\left|\frac{e^y}{y^n} - 0\right| \geq 947 = \varepsilon.$

This completes the proof of <u>non-uniform</u> convergence $f_n \rightrightarrows 0$, and therefore of uniform convergence to <u>any</u> function. $\blacksquare$

<u>Pf of Subclaim</u>: Induction on $n$ + L'Hopital's. $\blacksquare$

Assignment 4, Q4

4. Consider the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}.$$

Which of the following statements are true about this series?

- ☐ does not converge for any $x \in \mathbb{R}$; False ($x=0$)
- ☐ converges pointwise on a non-empty set but not on all of $\mathbb{R}$; $\hookrightarrow$ False
- ☐ converges pointwise on $\mathbb{R}$;

$\hookrightarrow$ True



- ☐ converges uniformly on a non-empty set but not on all of $\mathbb{R}$;
- ☐ converges uniformly on $\mathbb{R}$.

Sol'n:

$$f(x) = \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

$$x=0 \Rightarrow \sum_{n=1}^{\infty} \frac{0}{n(1+n(0^2))} = 0+0+0+0+\dots$$

$$= 0.$$

At $x=0$, we have pointwise convergence.

Now $x \neq 0$.

$$\frac{x}{n(1+nx^2)} = \frac{x}{n+n^2x^2} = \frac{1}{n^2} \frac{x}{\left(\frac{1}{n}+x^2\right)}$$

$$\leq \frac{1}{n^2} \cdot \frac{x}{(0+x^2)}$$

$$= \frac{1}{n^2} \cdot \frac{\cancel{x}}{x^{\cancel{2}}} = \frac{1}{n^2} \cdot \frac{1}{x}.$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \frac{1}{x}$

$$= \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges to $\frac{\pi^2}{6}$

(p-series)

(integral test)

$\therefore \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges pointwise on $\mathbb{R}$.

$(x = 0,\ x \neq 0)$

<u>Claim</u>: $\{f_n\}$ converges uniformly on <u>all</u> of $\mathbb{R}$.

<u>Proof</u>: Let $g_n(x) = \frac{x}{n(1+nx^2)}$.

Then $\lim g_n(x) = 0 = \lim g_n(-x)$.

$x \to -\infty$ $x \to +\infty$

So, $g_n(x)$ achieves its sup/inf at some real numbers $s_i$, $i \in \mathbb{R}$.

Set $g_n'(x_0) = 0$, solve for $x_0$.

Get: $x_0 = \frac{1}{\pm\sqrt{n}}$.

So sup/inf of $g_n$ are achieved at $\frac{1}{-\sqrt{n}}$, $\frac{1}{+\sqrt{n}}$. Plug into $g_n$ to find sup/inf.

$$g_n\left(-\frac{1}{\sqrt{n}}\right) = \frac{-1/\sqrt{n}}{n\left(1+n\left(-\frac{1}{\sqrt{n}}\right)^2\right)}$$

$$= \frac{-1/\sqrt{n}}{n(1+n\frac{1}{n})} = -\frac{1/\sqrt{n}}{2n}$$

$$= -\frac{n^{-1/2}}{2n} = -\frac{n^{-\frac{1}{2}-1}}{2}$$

$$= -\frac{n^{-3/2}}{2}.$$

Similar: $g_n\left(\frac{1}{\sqrt{n}}\right) = \frac{n^{-3/2}}{2}$.

Conclusion: $|g_n(x)| \leq \frac{1}{2} n^{-3/2}$

$\forall x \in \mathbb{R}$.

$$\left(\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)} = \sum_n g_n(x)\right)$$

But $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2} n^{-3/2}$ converges

(p-series OR integral test)

CONCLUSION:

$\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges uniformly $\forall x \in \mathbb{R}$ by M-test.

$$M_n = \frac{1}{2} n^{-3/2}.$$