Mathematics 3A03 — Real Analysis I

TERM TEST 1 — 26 February 2026

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of **14 pages** (*i.e.,* **7 double-sided pages**). There are **6 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There are blank pages after questions 4, 5 and 6, and additional blank pages at the end.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

$f : [-1, 1] \to \mathbb{R}$

(a) Every integrable function is continuous.

TRUE **FALSE**

$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$

(b) If f is integrable on $[a, b]$ and $F(x) = \int_a^x f$ then F has a maximum and minimum value on $[a, b]$.

1. Integrals are cts.

TRUE **FALSE** (F is cts.)

2. F attains its max & min by EVT.

(c) The instructor for this course is Bad Bunny.

TRUE **FALSE**

(d) If E has no accumulation points, then E is not closed.

TRUE **FALSE** $\varnothing$

(e) If E is open, then $\partial E \cap E = \varnothing$.

E open $\Rightarrow E = \mathring{E}$.

TRUE **FALSE** But in general, $\mathring{E} \cap \partial E = \varnothing$.

(f) If $E \subset \mathbb{R}$ and there is a function $f : E \to \mathbb{R}$ that is locally bounded on E then E is compact.

$f : \mathbb{R} \to \mathbb{R}$.

TRUE **FALSE**

$f(x) = 0$.

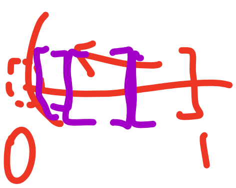
[9] **QUESTION 2.** For each of the sets E in the table below, answer **YES** or **NO** in each column to indicate whether or not E is open, closed, or compact. *Do not justify your answers.*

Set E	Open?	Closed?	Compact?
$[0,\infty)$	N	Y	N
$\bigcup_{n=1}^{\infty}\left[\frac{1}{n+1},\frac{1}{n}\right]$	N	N	N
$\bigcap_{n=1}^{\infty}\left[\frac{1}{n+1},\frac{1}{n}\right]$	Y	Y	Y

$(0,1]$

$= (0,1]$

$\emptyset =$



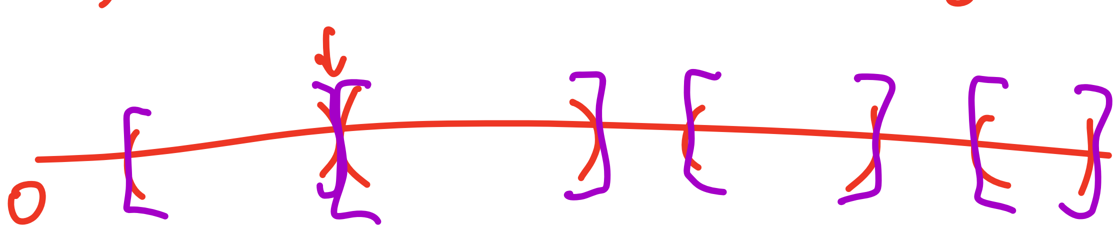
$$\left[\tfrac{1}{2},1\right]\cap\left[\tfrac{1}{3},\tfrac{1}{2}\right]\cap\left[\tfrac{1}{4},\tfrac{1}{3}\right]\dots=\emptyset$$

[6] **QUESTION 3.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the closure ($\overline{E}$), the interior (E°), and the boundary (∂E). *Do not justify your answers.*

density $\mathbb{Q}^c$ — density $\mathbb{Q}$ — density $\mathbb{Q}^c$, $\mathbb{Q}$

E	$\overline{E}$	E°	∂E
$[0,1]\cap\mathbb{Q}^{c}$	$[0,1]$	$\emptyset$	$[0,1]$
$\bigcup_{n=1}^{\infty}\left(n,n+\frac{1}{n}\right)$	$\bigcup_{n=1}^{\infty}\left[n,n+\frac{1}{n}\right]$	E	$\{1,2,3,\dots\}\cup\{1+\frac{1}{1},2+\frac{1}{2},3+\frac{1}{3},\dots\}$

$$\left(1,1+\tfrac{1}{1}\right)\cup\left(2,2+\tfrac{1}{2}\right)\cup\left(3,3+\tfrac{1}{3}\right)$$



[9] **QUESTION 4.**

[2] (a) State the formal definition of "the function f is *differentiable* at the point $c \in \mathbb{R}$".

f is differentiable at c if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{exists.}$$

[2] (b) State the *Mean Value Theorem* (MVT).

If f is cts. on $[a,b]$ and diff. on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

:)

[5] (c) Prove that $e^x \geq x+1$ for all $x \in \mathbb{R}$ by applying the Mean Value Theorem to $f(x) = e^x$.

Hint: Consider separately the cases $x = 0$, $x > 0$ and $x < 0$.

Case 1: $x=0$. Then $e^x = e^0 = 1 \geq 0+1 = x+1$. ✓

:)

Case 2: $x>0$. Apply MVT to $f(t)=e^t$ on $[0,x]$.

Then by MVT, there exists $c \in (0,x)$ such that

$f'(c) = \dfrac{e^x - e^0}{x-0}$. $\iff e^c = \dfrac{e^x-1}{x} \iff e^x = xe^c + 1$.

Now, $c>0$ and $f(t)=e^t$ is an increasing function.

So, $e^c \geq e^0$. i.e. $e^c \geq 1$.

Recall: $e^x = x\underbrace{e^c}_{\geq 1} + 1 \geq x(1) + 1 = x+1.$

i.e. $e^x \geq x+1.$:)

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Case 3: $x < 0$. Apply MVT to $f(t) = e^t$ on $[x, 0]$. Then $\exists c \in (x, 0)$ s.t. $f'(c) = \frac{e^0 - e^x}{0 - x}$.

$\iff e^c = \frac{1-e^x}{-x} \iff e^x = e^c x + 1$

But $c \leq 0$ and $f(t) = e^t$ is inc. So $e^c \leq e^0 = 1$.

$$\boxed{e^c \leq 1}$$

But $x < 0$, so $xe^c \geq x(1)$.

i.e. $xe^c \geq x \iff xe^c + 1 \geq x+1.$

In conclusion, $e^x = \underbrace{e^c x + 1}_{\geq x+1} \geq x+1$, as desired. :)

[10] **QUESTION 5.** Let $a < b$ and suppose f is a strictly increasing function defined on $[a, b]$.

[2] (a) Prove that f is bounded on $[a, b]$.

f is increasing, so $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$. So $f(a)$ is a lower bound and $f(b)$ is an upper bound.

[2] (b) Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Give the definition of the upper sum $U(f, P)$ and of the lower sum $L(f, P)$.

$$U(f,P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}),$$

$$L(f,P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}),$$

where $M_i := \sup_{t \in [t_{i-1}, t_i]} \{f(t)\}$ and

$m_i := \inf_{t \in [t_{i-1}, t_i]} \{f(t)\}.$

. . . Part (c) on next page. . .

[6] (c) Prove that f is integrable on $[a, b]$.

Hint: Consider an evenly spaced partition P, so $t_i - t_{i-1} = \delta$ for each i. Prove that

$$U(f, P) - L(f, P) \leq \delta[f(b) - f(a)]$$

and use this to show that f is integrable.

From part (a), we know f is bdd.

Recall the criterion: For f bdd, f is integrable on $[a,b]$ iff $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large s.t.

$$\frac{1}{N} < \frac{\varepsilon}{f(b)-f(a)}$$ (Archimedean Property of $\mathbb{R}$).

Let $P = \{x_0, \ldots, x_N\}$, where $x_i = a + \frac{i(b-a)}{N}$.

Then $U(f,P) - L(f,P) = \sum_i M_i (x_i - x_{i-1}) - \sum_i m_i (x_i - x_{i-1})$

f is increasing

$$= \sum_i f(x_i)(x_i - x_{i-1}) - \sum_i f(x_{i-1})(x_i - x_{i-1})$$

$M_i = f(x_i)$

$m_i = f(x_{i-1})$

$$= \sum_i \big(f(x_i) - f(x_{i-1})\big)(x_i - x_{i-1})$$

$$= \sum_i \left(f(x_i) - f(x_{i-1})\right) \cdot \frac{1}{N}$$

$$\left(a + \frac{i(b-a)}{N}\right) - \left(a + \frac{(i-1)(b-a)}{N}\right) = \frac{1}{N}$$

$$= -\frac{f(x_0)}{N} + \frac{f(x_N)}{N}$$

telescoping

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$$= -\frac{f(a)}{N} + \frac{f(b)}{N}$$

$$= \frac{1}{N}\left(f(b) - f(a)\right).$$

$$< \frac{\varepsilon}{f(b) - f(a)} \cdot \left(f(b) - f(a)\right) = \varepsilon \quad \checkmark \quad :)$$

...Continued...

[10] **QUESTION 6.** Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is integrable.

[5] (a) Prove that there exists $c \in [a,b]$ such that

$$\int_a^c f = \int_c^b f. \qquad (\heartsuit)$$

Hint: Define $F(x) = \int_a^x f$ and first show that ($\heartsuit$) can be expressed in term of F.

Note: When using theorems proved in class, state them clearly.

$F(x) = \int_a^x f.$

$$F(c) = F(b) - F(c)$$

$$\iff F(c) = \frac{F(b)}{2}.$$

Case 1: $F(b)=0$. Then $c=a$ or $c=b$ works.

Case 2: $F(b)>0$. Then F is cts by the integrals are cts THM. So F is subject to IVT. Moreover, $\underset{0}{\overset{\shortparallel}{F(a)}} < \frac{F(b)}{2} < F(b)$,

So, by IVT, $\exists c \in (a,b)$ s.t. $F(c) = \frac{F(b)}{2}$.

i.e. $\int_a^c f = \int_c^b f$.

. . . Part (b) on next page . . .

Case 3: $F(b) < 0$ similar.

[5] (b) Show, by constructing an example, that it is possible that c in part (a) might necessarily be an endpoint of the interval $[a, b]$. Specifically, take $a = 0$ and $b = 1$ and construct an integrable function f on $[0, 1]$ such that there is no $c \in (0, 1)$ for which (♡) holds but (♡) does hold for $c = 0$ and $c = 1$.

$f(x) = x - \frac{1}{2}$ on $[0, 1]$.

This works.

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