3A Tutorial 6

TA: Jeff

OH: 11:30 AM $\rightarrow$ 12:30 PM Mon
↳ HH 403 (or 4th floor study area)

Recordings + notes online!

Test this Thursday 7-10 PM!

Question 1: True or False?

(a) Every continuous function is integrable.

T. THM 8.1 TBB.

(b) If f is integrable on $[a,b]$, then for all

$0 \leq y \leq \int_a^b f$, there exists $a \leq c \leq b$ such that

$$\int_a^c f = y.$$

T. Integrals are cts. THM + IVT

(c) If every point in E is an accumulation point, then E is closed.

F. Take $E=(-1,1)$.

(d) If E is closed, then E is not open.

F. Take $E=\emptyset$ or $E=\mathbb{R}$.

(T if $E\neq\emptyset$ and $E\neq\mathbb{R}$.)

(e) If E is compact and $f: E\to\mathbb{R}$ is continuous, then f is bounded.

T. Continuous image of a compact set

is compact. Compact $\not\Rightarrow$ bounded.

Question 2: Yes or no?

Set E	Open?	Closed?	Compact?
$\mathbb{Q}$	N	N	N
$\bigcup_{n=1}^{\infty}(-2^{-\frac{1}{n}}, 2^{\frac{1}{n}})$	Y	N	N
$\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$	N	Y	Y

Question 3: Fill in the blank!

E	$\overline{E}$	E°	∂E
$\bigcup_{n=1}^{\infty}\{\frac{1}{n}\}$	$\{0\}\cup E$	$\emptyset$	$\{0\}\cup E$
$\bigcap_{n=1}^{\infty}[\frac{1}{n+1}, \frac{1}{n}]$	$\emptyset$	$\emptyset$	$\emptyset$

Question 4 (Definitions and statements)

(a) Define what it means for $f:\mathbb{R}\to\mathbb{R}$ to be integrable on $[a,b]$ (where $a<b$).

There exists $I\in\mathbb{R}$ such that for every $\varepsilon>0$, there is some $\delta>0$ with the property that whenever

$a=x_0\le t_1\le x_1\le t_2\le x_2\le\cdots\le x_{n-1}\le t_n\le x_n=b$

is a tagged partition satisfying $\max\limits_{1\le i\le n}\{x_i-x_{i-1}\}<\delta$, we have

$$\left|I-\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})\right|<\varepsilon.$$

(b) State the "integrals are continuous" THM.

If f is integrable on $[a,b]$ and $F(x)=\int_a^x f(t)\,dt$, then F is continuous.

(c) Prove that $\frac{a-1}{a}\le \ln(a)\le a-1$ for all $a>1$.

MVT applied to $\ln(x)$ on $[1, a]$.

$\exists\, c \in [1, a]$ s.t. $\ln'(c) = \dfrac{\ln(a) - \ln(1)}{a - 1}$

$$\Leftrightarrow \frac{1}{c} = \frac{\ln(a)}{a-1}.$$

But $1 \leq c \leq a \Rightarrow \frac{1}{a} \leq \frac{1}{c} \leq 1$

$$\Rightarrow \frac{1}{a} \leq \frac{\ln(a)}{a-1} \leq 1 \Rightarrow \frac{a-1}{a} \leq \ln(a) \leq a - 1.$$

Question 5

(a) Fill in the blank: f is integrable on $[a,b]$ if and only if f is bdd and for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that...

$$U(f,P) - L(f,P) < \varepsilon.$$

(b) Prove that $L(f,P) \leq U(f,P)$.

Write P as $a = x_0 < x_1 < \cdots < x_n = b$.

Then $m_i = \inf_{x_{i-1} \le t \le x_i} \{f(t)\}$, $M_i = \sup_{x_{i-1} \le t \le x_i} \{f(t)\}$.

So $m_i \le M_i$. Finally,

$$L(f,P) = \sum_i m_i (x_i - x_{i-1}) \le \sum_i M_i (x_i - x_{i-1})$$
$$= U(f,P).$$

(c) Suppose there are partitions P_0 and P_1 of $[a,b]$ such that $U(f,P_1) \le L(f,P_0)$. Prove that f is integrable.

Hint: Consider the union $P' = P_0 \cup P_1$. Then

$L(f,P_0) \le L(f,P')$ and $U(f,P') \le U(f,P_1)$

because P' is a refinement of both P_0 and P_1 (prove this). Conclude using parts (a) and (b) above.

BS 7.3

14. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

Let $F(x) = \int_0^x f$. By FTC, F is diff. and, $F' = f$, and $\int_a^b f = F(b) - F(a)$.

Let $c \in [0,1]$. Then $F(c) = \int_0^c f = F(c) - F(0)$.

But also, $\int_0^c f = \int_c^1 f = F(1) - F(c)$.

So, $F(c) - F(0) = F(1) - F(c)$.

$\Rightarrow 2F(c) = F(1) - F(0).$

$\Rightarrow F(c) = \dfrac{F(1) - F(0)}{2}.$ So,

F is constant. But $f = F' \Rightarrow f \equiv 0.$

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Question 1: True or False?

(a) Every continuous function on $[a,b]$ is integrable.

T

Lebesgue's criterion

$f(x) \geq 0 \quad \forall x$

(b) If f is integrable on $[a,b]$, then for all

$0 \leq y \leq \int_a^b f$, there exists $a \leq c \leq b$ such that

$\int_a^c f = y$.

T. S's are cts. + IVT.

(c) If every point in E is an accumulation point, then E is closed. False. $(0,1)$.

(d) If E is closed, then E is not open.

F. $E=\emptyset$, $E=\mathbb{R}$.

(T if $\emptyset \neq E \subsetneq \mathbb{R}$).

(e) If E is compact and $f: E \to \mathbb{R}$ is continuous, then f is bounded. T. EVT.

Question 2: Yes or no?

Set E	Open?	Closed?	Compact?
$\mathbb{Q}$	N	N	N
$\bigcup_{n=1}^{\infty}(-2^{-\frac{1}{n}}, 2^{\frac{1}{n}})$	Y	N	N
$\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$	N	Y	Y

Question 3: Fill in the blank!

E	$\overline{E}$	E°	∂E
$\bigcup_{n=1}^{\infty}\{\frac{1}{n}\}$	$E \cup \{0\}$	$\varnothing$	$E \cup \{0\}$
$\varnothing = \bigcap_{n=1}^{\infty}[\frac{1}{n+1}, \frac{1}{n}]$	$\varnothing$	$\varnothing$	$\varnothing$

Question 4 (Definitions and statements)

(a) Define what it means for $f: \mathbb{R} \to \mathbb{R}$ to be integrable on $[a, b]$ (where $a < b$).

Flesh it out: $U(f, P) - L(f, P) < \varepsilon$

Flesh it out: $\overline{\int} = \underline{\int}$

Flesh it out: tagged partitions

$$\sum f(t_i) \Delta_i.$$

(b) State the "integrals are continuous" THM.

Let f be int. on $[a, b]$. Let

$F(x) = \int_a^x f(t)\,dt$ $(a \leq x \leq b)$. Then F cts.

(c) Prove that $\frac{a-1}{a} \leq \ln(a) \leq a - 1$ for all $a > 1$.

MVT to $\ln$ on $[1, a]$

Solution 1: $\ln(a) = \int_1^a \frac{1}{t}\,dt$.

For $1 \le t \le a$, $\frac{1}{a} \le \frac{1}{t} \le 1$.

So, $\left(\frac{1}{a}\right)\underbrace{(a-1)}_{\text{width.}} \le \int_1^a \frac{1}{t}\,dt \le (1)\underbrace{(a-1)}_{\text{width}}$

$\Leftrightarrow \quad \frac{a-1}{a} \le \ln(a) \le a-1$.

Question 5

(a) Fill in the blank: f is integrable on $[a,b]$ if and only if, for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that...

f is $\quad U(f,P) - L(f,P) < \varepsilon$.

bold AND

(b) Prove that $L(f,P) \le U(f,P)$.

(c) Suppose there are partitions P_0 and P_1 of $[a, b]$ such that $U(f, P_1) \le L(f, P_0)$. Prove that f is integrable. f bdd.

(1) $\overline{\int} = \underline{\int}$

(2) Common refinement.

BS 7.3

14. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

Let $F(x) = \int_0^x f(t)\,dt$. By FTC, $F'(x) = f(x)$.

LHS: $F(x)$.

segmentation of the $\int$ OR FTC

RHS: $\int_x^1 f(t)\,dt = \int_0^1 f(t)\,dt - \int_0^x f(t)\,dt$

$= F(1) - F(x)$.

LHS=RHS: $F(x) = F(1) - F(x)$

$\Leftrightarrow \quad F(x) = \frac{F(1)}{2}$ does not depend on x.

i.e. F is constant. So, $F' = 0$.

But recall $F' = f$ (FTC). So, $f = 0$.

$\blacksquare$