3A Tutorial 5

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OH: Mon 11:30AM – 12:30PM
↳ HH 4th floor study area
(or HH 410).

Tutorial notes + video recordings online

BS §8.3

5. If $x \geq 0$ and $n \in \mathbb{N}$, show that

$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1} + \frac{(-x)^n}{1+x}.$$

Use this to show that

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}\,dt$$

and that

$$\left|\ln(x+1) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| \leq \frac{x^{n+1}}{n+1}.$$

6. Use the formula in the preceding exercise to calculate ln 1.1 and ln 1.4 accurate to four decimal places. How large must one choose n in this inequality to calculate ln 2 accurate to four decimal places?

<u>Sol'n to 5</u>: Let

$$S = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1}.$$

Then $S = 1 - x\left(1 - x + x^3 + \cdots + (-x)^{n-2}\right)$

$$= 1 - x\left(S - (-x)^{n-1}\right)$$

$$\Rightarrow S = 1 - xS - (-x)^n$$

$$\Rightarrow S(1+x) = 1-(-x)^n$$

$$\Rightarrow S = \frac{1}{1+x} - \frac{-(x)^n}{1+x}$$

$$\Rightarrow \frac{1}{1+x} = S + \frac{(-x)^n}{1+x}$$

$$\Rightarrow \frac{1}{1+x} = 1-x+x^2-x^3+\cdots+(-x)^{n-1} + \frac{(-x)^n}{1+x}.$$

First part done.

$$\frac{1}{1+t} = 1-t+t^2-t^3+\cdots+(-t)^{n-1} + \frac{(-t)^n}{1+t}$$

$$\Rightarrow \int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(1-t+\cdots+(-t)^{n-1}\right)dt + \int_0^x \frac{(-t)^n}{1+t}\,dt$$

$$\Rightarrow \ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+\frac{(-x)^n}{n}+\int_0^x \frac{(-t)^n}{1+t}\,dt.$$

Second part done.

$$\left|\frac{(-t)^n}{1+t}\right| = \frac{t^n}{1+t} \le t^n$$

$t \geq 0$

$$\Rightarrow \int_0^x \left|\frac{(-t)^n}{1+t}\right| dt \le \int_0^x t^n \, dt = \frac{x^{n+1}}{n+1}$$

So,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-x)^n}{n} + \int_0^x \frac{(-t)^n}{1+t} dt$$

$$\Rightarrow \ln(1+x) - \left(x - \frac{x^2}{2} + \cdots + \frac{(-x)^n}{n}\right) = \int_0^x \frac{(-t)^n}{1+t} dt$$

$$\Rightarrow \left|\ln(1+x) - \left(x - \frac{x^2}{2} + \cdots + \frac{(-x)^n}{n}\right)\right| \le \left|\int_0^x \frac{(-t)^n}{1+t} dt\right|$$

$$\le \int_0^x \left|\frac{(-t)^n}{1+t}\right| dt \le \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \left| \ln(1+x) - \left(x - \frac{x^2}{2} + \dots + \frac{(-x)^n}{n}\right) \right| \le \frac{x^{n+1}}{n+1}.$$

Last part done!

<u>Sol'n to 6</u>: Let's just do the $\ln(1.1)$ part.

Want: $\dfrac{0.1^{n+1}}{n+1} \le 10^{-5}$

$$\Leftarrow 0.1^{n+1} \le 10^{-5}$$

$$\Leftrightarrow \left(\frac{1}{10}\right)^{n+1} \le \left(\frac{1}{10}\right)^{5}$$

$$\Leftrightarrow n+1 \geq 5$$

$$\Leftrightarrow n \geq 4.$$

So, $\ln(1.1) \approx 0.1 - \frac{(0.1^2)}{2} + \frac{(0.1^3)}{3} - \frac{(0.1^4)}{4}$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4}$$

$$\begin{aligned} = &\ 0.1 \\ - &\ 0.005 \\ + &\ 0.000333333\ldots \\ - &\ 0.000025 \end{aligned}$$

$$= 0.0953083333\ldots$$

$$\Rightarrow \ln(1.1) = 0.\underbrace{0953}_{\text{first 4 decimals}}$$

Try $\ln(1.4)$ (harder). hint: $n=4$ works (check!)

Last part of the question: How many terms to compute $\ln(2)$ to 4 decimals?

$$\frac{\left(1^{n+1}\right)}{n+1} \leq 10^{-5} \iff \frac{1}{n+1} \leq 10^{-5}$$

$$\iff n+1 \geq 10^{5}$$

$$\iff n \geq 10^{5} - 1$$

$$= 100000 - 1$$

$$= 99999.$$

That's a lot!

TBB

4.2.2 Give an example of each of the following or explain why you think such a set could not exist.

(a) A nonempty set with no accumulation points and no isolated points

(b) A nonempty set with no interior points and no isolated points

(c) A nonempty set with no boundary points and no isolated points

Soln:

(a) No such set exists.

Assume for $\exists\ \phi \neq X \subseteq \mathbb{R}$ has no isolated points <u>and</u> no accumulation points.

Let $y \in X$. Then y is <u>not</u> an accumulation point, so $\exists$ an open set $U \subseteq \mathbb{R}$ such that $U \cap X = \{y\}$.

But this implies y is an isolated point — contradiction.

(b) $X = \mathbb{Q}$ is such a set.

<u>Non-empty</u>: $67 \in \mathbb{Q}$.

No interior points: Let $x \in \mathbb{Q}$ and let $U \subseteq \mathbb{R}$ be an open set containing x. Then $\exists \delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$ (definition of open).

By the density of irrational numbers, $\exists$ an irrational number y such that

$x-\delta < y < x+\delta$.

So, $U \not\subseteq \mathbb{Q}$. Since U was an arbitrary open set containing $x \in \mathbb{Q}$, we conclude that x is not an interior point of $\mathbb{Q}$.

No isolated points: Similar.

(c) $X = \mathbb{R}$ is an example.

Non-empty: $420 \in \mathbb{R}$.

No boundary points: $\forall\, x \in \mathbb{R}$ and open set $U \subseteq \mathbb{R}$ containing x, we have $U \cap \mathbb{R}^c = U \cap \emptyset = \emptyset$, so x is not a ∂ point.

No isolated points: Let $x \in \mathbb{R}$. Let $U \subseteq \mathbb{R}$ be an open set containing x. Then $\exists\, \delta > 0$ s.t. $(x-\delta, x+\delta) \subseteq U$.

So, $x + \frac{\delta}{2} \in U \setminus \{x\} \implies x$ not an isolated point.

§8.3 BS

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Use this to show that

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}\,dt$$

and that

$$\left|\ln(x+1) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| \leq \frac{x^{n+1}}{n+1}.$$

6. Use the formula in the preceding exercise to calculate ln 1.1 and ln 1.4 accurate to four decimal places. How large must one choose n in this inequality to calculate ln 2 accurate to four decimal places?

Sol'n to 5: Let $S = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1}$.

Then $S = 1 - x\left(1 - x + x^2 - \cdots + (-x)^{n-2}\right)$

$\Rightarrow S = 1 - x\left(1 - x + x^2 - \cdots + (-x)^{n-2} + (-x)^{n-1} - (-x)^{n-1}\right)$

$\Rightarrow S = 1 - x\left(S - (-x)^{n-1}\right)$

$$\Rightarrow S = 1 - xS - (-x)^n$$

$$\Rightarrow S + xS = 1 - (-x)^n$$

$$\Rightarrow S(1+x) = 1 - (-x)^n$$

$$\Rightarrow S = \frac{1}{1+x} - \frac{(-x)^n}{1+x}$$

$$\Rightarrow \frac{1}{1+x} = S - \frac{(-x)^n}{1+x}$$

$$\Rightarrow \frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^{n-1} - \frac{(-x)^n}{1+x}$$

$$x \mapsto t$$

$$\Rightarrow \frac{1}{1+t} = 1 - t + t^2 - \cdots + (-t)^{n-1} - \frac{(-t)^n}{1+t}$$

$$\Rightarrow \frac{1}{1+t} = \left(\sum_{i=0}^{n-1} (-t)^i\right) - \frac{(-t)^n}{1+t}$$

$$\Rightarrow \int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(\left(\sum_i (-t)^i\right) - \frac{(-t)^n}{1+t}\right) dt$$

Note: $\frac{d}{dt}\ln(1+t) = \frac{1}{1+t}$.

So, by FTC, LHS $= \ln(1+x)$.

$$\overset{\text{FTC}}{\Longrightarrow} \ln(1+x) = \underbrace{\left(\sum_i \int_0^x (-t)^i\,dt\right) - \int_0^x \frac{(-t)^n}{1+t}\,dt}_{\text{linearity of } \int}$$

$$\overset{\substack{\text{Power}\\\text{rule}\\+\\\text{FTC}}}{=} \left(\sum_i (-1)^i \left[\frac{t^{i+1}}{i+1}\right]_0^x\right) - \int_0^x \frac{(-t)^n}{1+t}\,dt$$

$$= \left(\sum_i (-1)^i \frac{x^{i+1}}{i+1}\right) - \int_0^x \frac{(-t)^n}{1+t}\,dt.$$

In sum:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{(-x)^n}{n} + \int_0^x \frac{(-t)^n}{1+t}\,dt$$

$$\Rightarrow \ln(1+x) - \left(x - \frac{x^2}{2} + \cdots - \frac{(-x)^n}{n}\right) = \int_0^x \frac{(-t)^n}{1+t}\,dt$$

$$\Rightarrow \left|\ln(1+x) - \left(x - \frac{x^2}{2} + \cdots - \frac{(-x)^n}{n}\right)\right| = \left|\int_0^x \frac{(-t)^n}{1+t}\,dt\right|$$

$$\leq \int_0^x \left|\frac{(-t)^n}{1+t}\right| dt \underset{t \geq 0}{=} \int_0^x \frac{t^n}{1+t}\,dt \leq \int_0^x t^n\,dt \overset{\text{FTC}}{=} \frac{x^{n+1}}{n+1}$$

$$\frac{t^n}{1+t} \leq t^n$$

$$\Rightarrow \left|\ln(1+x) - \left(x - \frac{x^2}{2} + \cdots - \frac{(-x)^n}{n}\right)\right| \leq \frac{x^{n+1}}{n+1}$$

Problem 5 QED.

Sol'n to 6: $\ln(1.1) = ?$ 4 D places Know:

$$\left|\ln(1+x) - \left(x - \frac{x^2}{2} + \cdots - \frac{(-x)^n}{n}\right)\right| \leq \frac{x^{n+1}}{n+1} \underset{\text{WANT}}{\leq} 5 \times 10^{-5}$$

Degenerate case: 1 0.9999999

just make sure our estimate doesn't end in 9.

$\Uparrow$

$\le 10^{-5}$

Want: (1) $\ln(1+x) = \ln(1.1)$

(2) $\dfrac{x^{n+1}}{n+1} \le 10^{-5}$

(1) $\Rightarrow x = 0.1$.

$x = 0.1 \Rightarrow \dfrac{0.1^{n+1}}{n+1} \le 10^{-5}$?

<u>But</u> $\dfrac{0.1^{n+1}}{n+1} \le 0.1^{n+1}$.

So $0.1^{n+1} \le 10^{-5}$ suffices.

$\Leftrightarrow (10)^{-(n+1)} \le 10^{-5}$

$\Leftrightarrow n+1 \geq 5 \Leftrightarrow n \geq 4$

So TAKE $n=4$. :)

Put is all together: $x=0.1$, $n=4$

$$\left|\ln(1.1)-\left(0.1-\frac{0.1^2}{2}+\frac{0.1^3}{3}-\frac{0.1^4}{4}\right)\right|\le 10^{-5}$$

So just compute $0.1-\frac{0.1^2}{2}+\frac{0.1^3}{3}-\frac{0.1^4}{4}$.

$$=10^{-1}-\frac{10^{-2}}{2}+\frac{10^{-3}}{3}-\frac{10^{-4}}{4}$$

$$=0.1-\frac{0.01}{2}+\frac{0.001}{3}-\frac{0.0001}{4}$$

$= 0.1$
$- 0.005$
$+ 0.000333333\ldots$
$- 0.000025$

$= 0.0953083333\ldots$

no 9 after 0.0953, so first 4 decimals

CONCLUSION: are correct.

$\ln(1.1) \approx \underbrace{0.0953}_{\text{these guys are correct}}\ldots$

$\ln(1.4)$ to 4 D places.

$n = 4.$

Last part $\ln(2)$:

$\ln(x+1) = \ln(2)$, $x = 1$

WANT $\dfrac{x^{n+1}}{n+1} \leq 10^{-5}$

$\Leftrightarrow \dfrac{1^{n+1}}{n+1} \leq 10^{-5}$

$\Leftrightarrow \dfrac{1}{n+1} \leq 10^{-5}$

$$\Leftrightarrow \quad n+1 \geq 10^5$$

$$\Leftrightarrow \quad n \geq 10^5 - 1$$

$$= 100000 - 1$$

$$= 99999$$