3A Tutorial 4

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BS 7.2.

10. If f and g are continuous on $[a, b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Sol'n: Proof by contraposition:

$$f(c) \neq g(c) \ \forall c \in [a,b] \implies \int_a^b f \neq \int_a^b g.$$

Indeed, if $f(c) \neq g(c)$ $\forall c \in [a,b]$, then

$f(a) < g(a)$ or $f(a) > g(a)$.

WLOG, assume $f(a) < g(a)$. We claim

$f(c) < g(x)$ $\forall c \in [a,b]$.

To see this, assume we have $a < c \le b$ s.t. $f(c) \ge g(c)$. If $f(c) = g(c)$, this contradicts our assumption that $f(x) \ne g(x)$ $\forall x \in [a,b]$. So, $f(c) > g(c)$.

Let $h(x) = f(x) - g(x)$. Then

$h(a) = f(a) - g(a) < 0$, while $h(b) > 0$.

Moreover, h is cts. as a linear combo. of cts. functions. It follows then, by the IVT, that $h(c) = 0$ for some $c \in [a,b]$. But this $\Rightarrow f(c) = g(c)$ —

again contradicting our assumption.

We conclude that $f(x) < g(x)$ $\forall x \in [a,b]$.

So, the function $h(x) = f(x) - g(x)$

satisfies $h(x) < 0$ $\forall x \in [a,b]$. By EVT,

$\exists c \in [a,b]$ s.t. $h(x) \leq h(c)$ $\forall x \in [a,b]$.

We therefore have the elementary bound

$$\int_a^b h(x)\,dx \leq h(c)(b-a) < 0.$$

But $\int_a^b h(x)\,dx = \int_a^b (f-g) = \left(\int_a^b f\right) - \left(\int_a^b g\right)$,

so $\int h < 0 \Rightarrow \int_a^b f < \int_a^b g$. In particular,

$\int_a^b f \neq \int_c^b g$, as desired.

BS 7.3

14. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

Let $F(x) = \int_0^x f$. By FTC, F is diff. and, $F' = f$, and $\int_a^b f = F(b) - F(a)$.

Let $c \in [0,1]$. Then $F(c) = \int_0^c f = F(c) - F(0)$.

But also, $\int_0^c f = \int_c^1 f = F(1) - F(c)$.

So, $F(c) - F(0) = F(1) - F(c)$.

$\Rightarrow 2F(c) = F(1) - F(0)$.

$\Rightarrow F(c) = \frac{F(1) - F(0)}{2}$. So,

F is constant. But $f = F' \Rightarrow f \equiv 0$.

BS 7.3

11. Find $F'(x)$ when F is defined on $[0, 1]$ by:

(a) $F(x) := \int_0^{x^2} (1 + t^3)^{-1}\, dt$.

(b) $F(x) := \int_{x^2}^{x} \sqrt{1 + t^2}\, dt$.

Sol'n: (a) Let $G(x) = \int_0^x (1+t^3)^{-1} dt$

and $h(x) = x^2$. Then

$F = G \circ h \Rightarrow F'(x) = h'(x)\, G'(h(x))$

$= 2x \frac{1}{1 + x^6}$.

(b) Let $G(x) = \int_0^x \sqrt{1+t^2}\, dt$.

Then $F(x) = G(x) - G(x^2)$. Let

$h(x) = x^2$. Then

$$F(x) = G(x) - G(h(x))$$

$$\Rightarrow F'(x) = G'(x) - h'(x)\, G'(h(x))$$

$$= \sqrt{1+x^2} - 2x\sqrt{1+x^4}.$$

BS 7.2.

18. Let f be continuous on $[a, b]$, let $f(x) \geq 0$ for $x \in [a, b]$, and let $M_n := \left(\int_a^b f^n\right)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a, b]\}$.

Sol'n: By the EVT, $\sup\limits_{a \le x \le b}\{f(x)\}$ is achieved by some $x \in [a,b]$. That is,

$f(c) = \sup\limits_{a \le x \le b}\{f(x)\}$ for some $c \in [a,b]$.

For $n \geq 1$, the map $y \longmapsto y^n$ is increasing

$(y \geq 0)$, so $f(c)^n = \sup\limits_{a \leq x \leq b} \{f(x)^n\}$.

We therefore have the elementary bound

$$\int_a^b f(x)^n \, dx \leq (b-a) f(c)^n$$

$$\Rightarrow \left(\int_a^b f(x)^n \, dx \right)^{\frac{1}{n}} \leq \left((b-a) f(c)^n \right)^{1/n}$$

$$\Rightarrow \boxed{M_n \leq (b-a)^{1/n} f(c)}$$

Next, let $\varepsilon > 0$. By continuity of

f, $\exists \, \delta > 0$ s.t. if $x \in [a,b]$ satisfies

$|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$

(in particular, $f(x) > f(c) - \varepsilon$.

Let $a' = \max\{a, c-\delta\}$, $b' = \min\{b, c+\delta\}$.

Then $\displaystyle\int_a^b f(x)^n dx \geq \int_{a'}^{b'} f(x)^n dx$

$$\geq (f(c) - \varepsilon)^n (b' - a')$$

$\Rightarrow (f(c) - \varepsilon)(b' - a')^{\frac{1}{n}} \leq M_n.$

$\Rightarrow (f(c) - \varepsilon)(b' - a')^{\frac{1}{n}} \leq M_n \leq f(c)(b-a)^{\frac{1}{n}}$

$$\xrightarrow[\text{Squeeze!!!}]{n \to \infty}$$

$f(c) - \varepsilon \leq \liminf M_n \leq \limsup M_n \leq f(c).$

But $\varepsilon > 0$ was arbitrary, so

$$f(c) \leq \liminf M_n \leq \limsup M_n \leq f(c).$$

$\Rightarrow \lim M_n$ exists and equals

$f(c) = \sup_{a \leq x \leq b} \{f(x)\}.$

BS §7.2.

10. If f and g are continuous on $[a, b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Sol'n: Contrapositive:

If $f(c) \neq g(c)$ $\forall c \in [a,b]$, then $\int_a^b f \neq \int_a^b g$.

Assume $f(c) \neq g(c)$ $\forall c \in [a,b]$. Then,

$f(a) < g(a)$ OR $f(a) > g(a)$. WLOG,

assume $f(a) < g(a)$.

Claim: $f(c) < g(c)$ $\forall c \in [a,b]$.

Sub pf: If $f(c) \geq g(c)$ for some

$c \in [a,b]$, then $f(c) > g(c)$ OR $f(c) = g(c)$,

But $f(c) \neq g(c)$ by assumption,

So, $f(c) > g(c)$.

Let $h(x) = f(x) - g(x)$. (h is cts. as a linear combination of cts. functions.

Moreover, $h(a) = f(a) - g(a) < 0$, while $h(c) = f(c) - g(c) > 0$. So, $h(x) = 0$ for some $x \in [a, c]$ (IVT).

This implies $f(x) = g(x)$ — again contradicting our assumption.

This proves the Claim subQED.

It follows that $h(x) = f(x) - g(x) < 0$ for all $x \in [a, b]$. By the extreme value THM, there exists $c \in [a, b]$

such that $h(c) \geq h(x)$ $\forall x \in [a,b]$.

We then have the elementary bound $\int_a^b h(x)\,dx \leq (b-a)h(c) < 0$.

But $\int_a^b h(x)\,dx = \int_a^b (f(x)-g(x))\,dx$

$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.

So $\int_a^b f < \int_a^b g$. In particular $\int_a^b f \neq \int_a^b g$. $\blacksquare$

BS §7.3

11. Find $F'(x)$ when F is defined on $[0, 1]$ by:

(a) $F(x) := \int_0^{x^2} (1+t^3)^{-1}\,dt$. (b) $F(x) := \int_{x^2}^{x} \sqrt{1+t^2}\,dt$.

Sol'n: (a) Let $G(x) = \int_0^x (1+t^3)^{-1}\,dt$. Let

$h(x) = x^2$. Then

$$F(x) = \int_0^{x^2} (1+t^3)^{-1}\,dt = G(x^2) = G(h(x)).$$

$$\Rightarrow F'(x) \underset{\substack{\uparrow \\ \text{Chain} \\ \text{rule}}}{=} h'(x)\,G'(h(x)) = 2x\,G'(x^2).$$

By FTC, $G'(x) = (1+x^3)^{-1}$. So,

$$F'(x) = 2x\,G'(x^2) = 2x\left(1+(x^2)^3\right)^{-1}$$

$$= 2x(1+x^6)^{-1}.$$

(b) Let $G(x) = \int_0^x \sqrt{1+t^2}\,dt$, $h(x) = x^2$.

Then $F(x) = \int_{x^2}^{x} \sqrt{1+t^2}\,dt$

$$= \left(\int_0^x \sqrt{1+t^2}\,dt\right) - \left(\int_0^{x^2} \sqrt{1+t^2}\,dt\right)$$

$$= G(x) - G(x^2) = G(x) - G(h(x)),$$

So, $F'(x) = G'(x) - (G \circ h)'(x)$

$$\underset{\text{FTC + Chain rule}}{=} \sqrt{1+x^2} - h'(x)\,G'(h(x))$$

$$\underset{\text{FTC}}{=} \sqrt{1+x^2} - 2x\sqrt{1+(x^2)^2}.$$

$$= \sqrt{1+x^2} - 2x\sqrt{1+x^4}.$$

BS §7.2.

18. Let f be continuous on $[a, b]$, let $f(x) \geq 0$ for $x \in [a, b]$, and let $M_n := \left(\int_a^b f^n\right)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a, b]\}$.

"the p-norms converge to sup-norm"

Sol'n: f is cts. and therefore R-integrable.

$f(x)^n = f(x) \cdot f(x) \cdots f(x)$ is R-integrable

$\underbrace{\qquad\qquad}_{n \text{ times}}$

as a product of R-integrable functions. So $\forall n$, $M_n = \left(\int_a^b f^n\right)^{1/n}$ is a well-defined real number.

By the Extreme Value THM (EVT), $\sup_{a \le x \le b} \{f(x)\}$ is achieved by some $c \in [a,b]$. That is, $f(c) = \sup_{a \le x \le b} \{f(x)\}$.

$$(\le) \quad M_n = \left(\int_a^b f^n\right)^{\frac{1}{n}} \le \left(f(c)^n (b-a)\right)^{\frac{1}{n}}$$

$f(c) = \sup_{a \le x \le b} \{f(x)\} \Rightarrow f(c)^n = \sup_{a \le x \le b} \{f(x)^n\}$.

$$\Rightarrow M_n \le f(c)(b-a)^{1/n}.$$

($\geq$) Let $\varepsilon > 0$. Since f is cts., there exists $\delta > 0$ such that if $x \in [a,b]$ satisfies $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

In particular, $f(x) > f(c) - \varepsilon$.

$|f(x) - f(c)| < \varepsilon \iff -\varepsilon < f(x) - f(c) < \varepsilon$

$\Rightarrow -\varepsilon < f(x) - f(c) \Rightarrow f(c) - \varepsilon < f(x)$.

Let $a' = \max\{a, c-\delta\}$, $b' = \min\{b, c+\delta\}$.

$\left(x \in [a', b'] \Rightarrow f(c) - \varepsilon < f(x)\right)$.

Then $M_n = \left(\int_a^b f^n\right)^{1/n} \underset{f(x) \geq 0}{\geq} \left(\int_{a'}^{b'} f^n\right)^{1/n}$

$$\geq \left((b'-a')(f(c)-\varepsilon)^{n} \right)^{1/n}$$

$$\Rightarrow M_n \geq (b'-a')^{1/n}(f(c)-\varepsilon).$$

Combining both inequalities:

$$(b'-a')^{1/n}(f(c)-\varepsilon) \leq M_n \leq (b-a)^{1/n} f(c).$$

$$\xrightarrow{n \to \infty}$$

$$f(c)-\varepsilon \leq \liminf M_n \leq \limsup M_n \leq f(c).$$

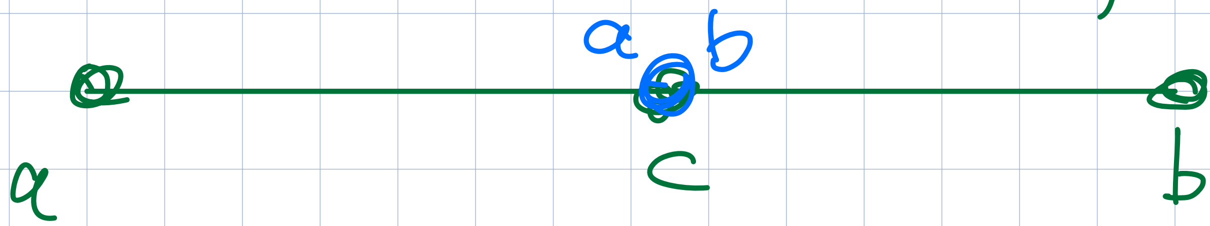
But $\varepsilon > 0$ was arbitrary. So in fact,

$$f(c) \leq \liminf M_n \leq \limsup M_n \leq f(c).$$

So, $\liminf M_n = \limsup M_n = f(c)$.

It follows that $\lim M_n$ exists and equals $f(c) = \sup_{a \leq x \leq b} \{f(x)\}$. ∎

$a \le c \le b \quad (a = b)$



$$F(x) = \int_0^{x^2} (1+t^3)^{-1}\,dt.$$

$$h(x) = x^2, \quad G(x) = \int_0^{x} (1+t^3)^{-1}\,dt$$

(FTC)

$$F(x) = G(h(x)) \overset{CR}{=} h'(x)\,G'(h(x))$$