3A03 Tutorial 3

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OH: Mon 11:30AM (HH 403)

Tutorial notes + video recordings posted online

TBB

8.2.9 Compute the Riemann sums for the integral $\int_a^b x^{-2}\,dx$ $(a > 0)$ taken over a partition

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$$

of the interval $[a, b]$ and with associated points $\xi_i = \sqrt{x_i x_{i-1}}$. What can you conclude from this?

Soln: $\displaystyle\sum_{i=0}^{n-1} f(\xi_i)\,\Delta_i$, $f(x) = x^{-2}$, $\xi_i = \sqrt{x_i x_{i-1}}$,

$\Delta_i = x_i - x_{i-1}$

$$\sum_{i=1}^{n} f(\xi_i)\Delta_i = \sum_i \xi_i^{-2}(x_i - x_{i-1})$$

$$= \sum_i \frac{1}{x_i x_{i-1}}(x_i - x_{i-1})$$

$$= \sum_i \frac{x_i}{x_i x_{i-1}} - \sum_i \frac{x_{i-1}}{x_i x_{i-1}}$$

$$= \sum_i \left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right)$$

$$= \left(\frac{1}{x_0} - \cancel{\frac{1}{x_1}}\right) + \left(\cancel{\frac{1}{x_1}} - \cancel{\frac{1}{x_2}}\right)$$

$$+ \left(\cancel{\frac{1}{x_2}} - \cancel{\frac{1}{x_3}}\right) \cancel{+} \dots$$

$$\cancel{+}\left(\cancel{\frac{1}{x_{n-1}}} - \frac{1}{x_n}\right) = \frac{1}{x_0} - \frac{1}{x_n}$$

$$= \frac{1}{a} - \frac{1}{b}$$

independent of n!

So, $\displaystyle\int_a^b \frac{1}{x^2}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta_i$

$$= \lim_{n\to\infty}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{1}{a} - \frac{1}{b}. \quad :)$$

Verify w/ FTC:

$$\int_a^b \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_a^b$$

$$= \left[-\frac{1}{b} - \left(-\frac{1}{a}\right)\right]$$

$$= \frac{1}{a} - \frac{1}{b} \quad :)$$

TBB :

8.2.13 Calculate

$$\lim_{n\to\infty} \frac{e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n}}{n}$$

by expressing this limit as a definite integral of some continuous function and then using calculus methods.

$$\frac{e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}} + e^{\frac{n}{n}}}{n}$$

$$\overset{\text{want}}{=} \sum_{i=1}^{n} f(\xi_i)\Delta_i \text{ , where}$$

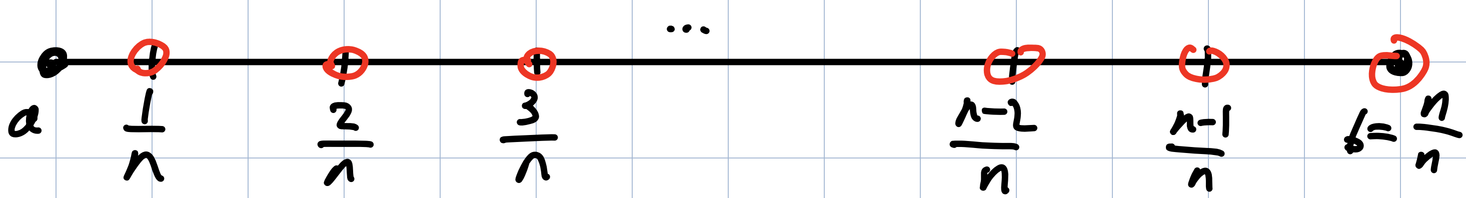
$$a = x_0 \le \xi_1 \le x_1 \le \cdots \le x_{n-1} \le \xi_n \le x_n = b$$

and $\Delta_i = x_i - x_{i-1}$

$$\frac{e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}} + e^{\frac{n}{n}}}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} e^{\frac{i}{n}}$$

$$= \sum_{i=1}^{n} \underbrace{e^{\frac{i}{n}}}_{f(\xi_i)} \cdot \underbrace{\frac{1}{n}}_{\Delta_i}$$



$$f(x) = e^x, \quad x_i = \frac{i}{n}, \quad \Delta_i = \frac{1}{n},$$

$$\xi_i = \frac{i}{n}$$

$$\frac{1}{n}\sum_{i=1}^{n} e^{\frac{i}{n}} \xrightarrow[n\to\infty]{} \int_0^1 e^x \, dx$$

$$\overset{\text{FTC}}{=} \left[e^x\right]_0^1 = e - 1.$$

TBB:

8.3.4 If f is continuous and nonnegative on an interval $[a,b]$ and

$$\int_a^b f(x)\,dx = 0$$

show that f is identically equal to zero there.

Sol'n: Assume for contradiction

$f(c) > 0$, some $c \in [a,b]$. Let

$\varepsilon = \frac{f(c)}{2}$. THEN $\exists \delta > 0$ s.t.

if $x \in [a,b]$ satisfies $|x-c| < \delta$,

then $|f(x) - f(c)| < \varepsilon = \frac{f(c)}{2}$

$$\Leftrightarrow$$

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}$$

$$\Rightarrow$$

$$f(c) - \frac{f(c)}{2} < f(x)$$

$$\Rightarrow$$

$$\frac{f(c)}{2} < f(x)$$

Let $a' = \max\{c-\delta, a\}$,

$b' = \min\{c+\delta, b\}$.

<u>Then</u>

$$\int_a^b f(x)\,dx = \int_a^{a'} f(x)\,dx + \int_{a'}^{b'} f(x)\,dx + \int_{b'}^{b} f(x)\,dx$$

$$\geq 0 + \int_{a'}^{b'} \frac{f(c)}{2}\,dx + 0$$

$$= \underbrace{\frac{f(c)}{2}}_{>0}\underbrace{(b'-a')}_{>0} > 0. \quad \lightning !!!$$

TBB:

8.2.2 Show that the number I in the statement of Theorem 8.1 is unique; that is, that there cannot be two numbers that would be assigned to the symbol $\int_a^b f(x)\,dx$.

Theorem 8.1 (Cauchy) *Let f be a continuous function on an interval $[a,b]$. Then there is a number I, called the* definite integral *of f on $[a,b]$, such that for each $\varepsilon > 0$ there is a $\delta > 0$ so that*

$$\left|\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - I\right| < \varepsilon$$

whenever $[x_0,x_1]$, $[x_1,x_2]$, $\ldots$, $[x_{n-1},x_n]$ is a partition of the interval $[a,b]$ into subintervals of length less than δ and each ξ_k is a point in the interval $[x_{k-1},x_k]$.

Sol'n: Assume for contradiction there exist two distinct real numbers $I, I' \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ with the property that for any tagged partition

$$a = x_0 \leq \xi_1 \leq x_1 \leq \cdots \leq x_{n-1} \leq \xi_n \leq x_n = b$$

with $\min\limits_{1 \leq i \leq n} \{x_i - x_{i-1}\}$, we have

$$\left|\sum_{i=1}^{n} f(\xi_i)\Delta_i - I\right| < \varepsilon \quad \underline{\text{and}}$$

$$\left|\sum_{i=1}^{n} f(\xi_i)\Delta_i - I'\right| < \varepsilon.$$

Let $\varepsilon = \frac{|I-I'|}{2}$. Let $\delta>0$ be the associated δ (see previous sentence). Let $a=x_0 \le \xi_1 \le x_1 \le \cdots \le x_{n-1} \le \xi_n \le x_n = b$ be any tagged partition of $[a,b]$ satisfying

$$\min_{1\le i\le n} \{x_i - x_{i-1}\} < \delta.$$

(For example, perhaps we take n so that $\frac{1}{n} < \delta$, and define $x_i = a + \frac{i}{n}$ and $\xi_i = x_i$).

Then, by construction of δ we have

$$\left| \sum_{i=1}^{n} f(\xi_i)\Delta_i - I \right| < \varepsilon \quad \underline{\text{and}}$$

$$\left| \sum_{i} f(\xi_i)\Delta_i - I' \right| < \varepsilon.$$

So, $|I - I'| \leq 2\varepsilon = 2\,\dfrac{|I-I'|}{2}$

$$= |I - I'|$$

— contradiction!

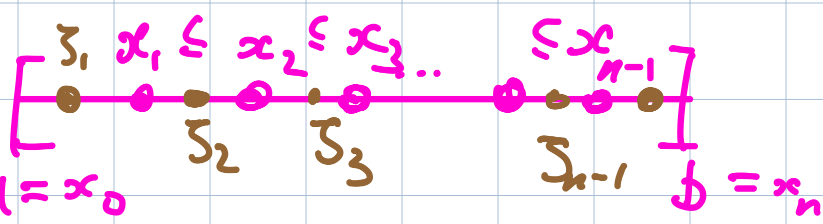
TBB

8.2.9 Compute the Riemann sums for the integral $\int_a^b x^{-2}\,dx$ $(a > 0)$ taken over a partition

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$$

of the interval $[a, b]$ and with associated points $\xi_i = \sqrt{x_i x_{i-1}}$. What can you conclude from this?

BS	TBB
Darboux $\int$	Riemann $\int$
Upper/lower sums	"tagged partitions"



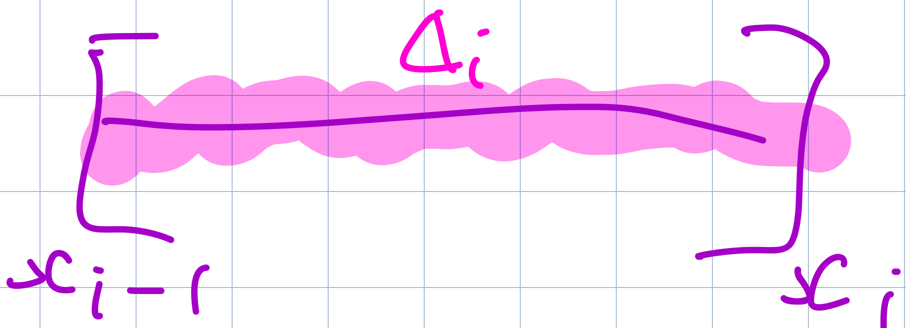
Sol'n: Riemann sum:

$$\sum_{i=1}^{n} f(\xi_i)\,\Delta_i$$

$f(x) = x^{-2}$

$\xi_i = \sqrt{x_i x_{i-1}}$

$\Delta_i = x_i - x_{i-1}$



Why $\xi_i \in [x_{i-1}, x_i]$?

Proof by Kyle: ξ_i is the geometric mean of x_i, x_{i-1}

$$\sum_i f(\xi_i)\,\Delta_i = \sum_i \xi_i^{-2}(x_i - x_{i-1})$$

$$= \sum_i \left(\cancel{\sqrt{x_i x_{i-1}}}\right)^{\cancel{-2}} (x_i - x_{i-1})$$

$$= \sum_i \frac{1}{x_i x_{i-1}} (x_i - x_{i-1})$$

$$= \sum_i \left(\frac{\cancel{x_i}}{\cancel{x_i} x_{i-1}} - \frac{\cancel{x_{i-1}}}{x_i \cancel{x_{i-1}}} \right)$$

$\boxed{a?0}$

$$= \sum_i \left(\frac{1}{x_{i-1}} - \frac{1}{x_i} \right)$$

Telescoping sam!

$$= \left(\frac{1}{x_0} - \cancel{\frac{1}{x_1}}\right) + \left(\cancel{\frac{1}{x_1}} - \cancel{\frac{1}{x_2}}\right)$$

$$+ \left(\cancel{\frac{1}{x_2}} - \cancel{\frac{1}{x_3}}\right) \cancel{+} \cdots \cancel{+} \left(\cancel{\frac{1}{x_{n-1}}} - \frac{1}{x_n}\right)$$

$$= \frac{1}{x_0} - \frac{1}{x_n} = \frac{1}{a} - \frac{1}{b}.$$

$$\int_a^b \frac{1}{x^2}\,dx = \lim_{\Delta_i \to 0} \sum f(\xi_i)\Delta_i$$

$$= \lim_{\Delta_i \to 0} \left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{1}{a} - \frac{1}{b}.$$

Check w/ calculus FTC

TBB

8.3.4 If f is continuous and nonnegative on an interval $[a,b]$ and

$$\int_a^b f(x)\,dx = 0$$

show that f is identically equal to zero there.

i.e. Prove $f(x)=0 \ \forall x \in [a,b]$.

Recall: Given $m, M \in \mathbb{R}$ s.t. $\forall x \in [a,b]$ we have $m \leq f(x) \leq M$, it follows that $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

(Continuity not needed!)

Soln: Proof by contraposition.

Assume $f(c) > 0$ for some $c \in [a,b]$. Conclude

$$\int_a f(x)dx \neq 0.$$

Let $c \in [a,b]$ be s.t. $f(c) > 0$. Let $\varepsilon = f(c)/2$ (note: $\varepsilon > 0$). By the continuity of f, there exists $\delta > 0$ s.t. if $x \in [a,b]$ satisfies $|x-c| < \delta$, then

$$|f(x) - f(c)| < \varepsilon = f(c)/2.$$

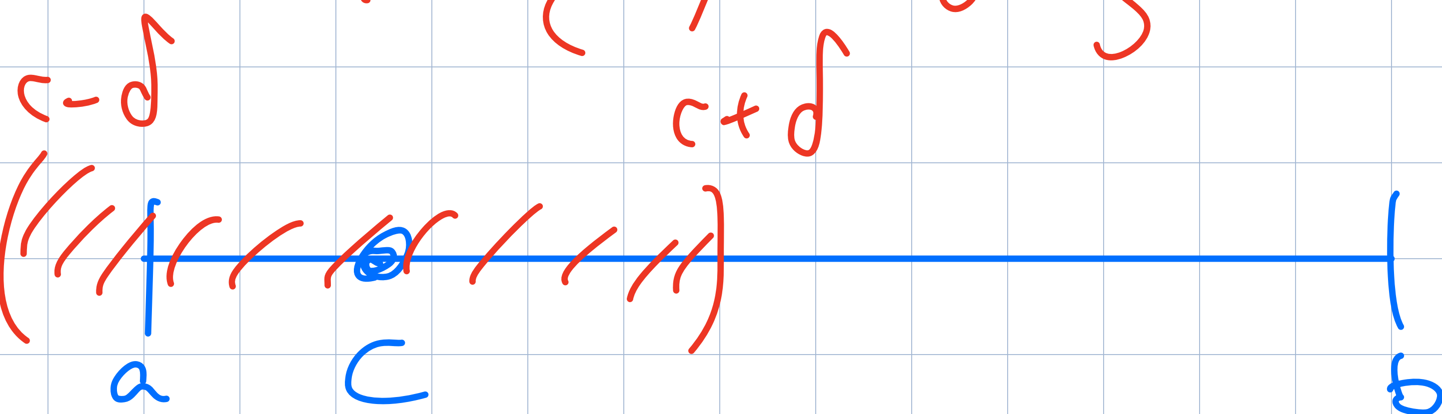
$$\Leftrightarrow -\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}$$

$$\Leftrightarrow f(c) - \frac{f(c)}{2} < f(x) < \frac{f(c)}{2} + f(c)$$

$$\Rightarrow \frac{f(c)}{2} < f(x). \quad \text{when } |x-c|<\delta.$$

Let $a' = \max\{a, c-\delta\}$,

$b' = \min\{b, c+\delta\}$.



Then $\frac{f(c)}{2} < f(x)$ when $x \in [a', b']$.

Then $$\int_a^b f(x)dx = \int_a^{a'} \underbrace{f(x)}_{\geq 0}dx + \int_{a'}^{b'} \underbrace{f(x)}_{> f(c)}dx + \int_{b'}^{b} \underbrace{f(x)}_{\geq 0}dx$$

$\overline{}$ 2 $\qquad\qquad$ >0

$$\geq (0)(a'-a) + \left(\frac{f(c)}{2}\right)(b'-a') + (0)(b-b')$$

$$= 0 + \underbrace{\frac{f(c)}{2}}_{>0}(b'-a') + 0$$

$$> 0.$$

So, $\int_a^b f(x)\,dx \neq 0$. ∎

$a=0,\quad b=1,\quad f(x)=\begin{cases}100, & x=\frac{1}{2}\\ 0, & \text{else.}\end{cases}$

$$\int_a^b f(x)\,dx = \heartsuit .$$

TBB

8.2.13 Calculate

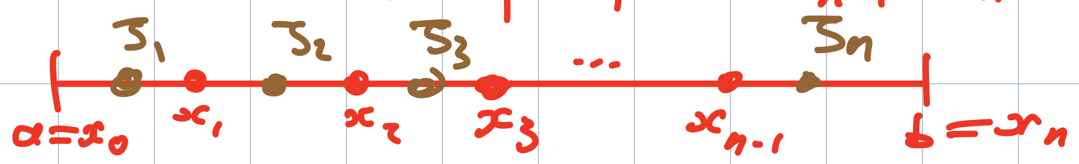
$$\lim_{n\to\infty} \frac{e^{1/n}+e^{2/n}+\cdots+e^{(n-1)/n}+e^{n/n}}{n}$$

by expressing this limit as a definite integral of some continuous function and then using calculus methods.

Sol'n: $\dfrac{e^{\frac{1}{n}}+e^{\frac{2}{n}}+\cdots+e^{\frac{n-1}{n}}+e^{\frac{n}{n}}}{n} = \sum_{i=1}^{n} \underbrace{\left(e^{\frac{i}{n}}\right)}_{f(\xi_i)}\cdot\underbrace{\left(\frac{1}{n}\right)}_{\Delta_i}$

$\int_a^b f(x)\,dx$. What are a and b?

What is f? What is $a=x_0\le \xi_1 \le x_1 \le \cdots \le x_{n-1} \le \xi_n \le x_n = b$.



Try:

$[a,b]=[0,1]$. $f(x)=e^x$. $x_i=\frac{i}{n}$. $\xi_i=x_i=\frac{i}{n}$.

$$\sum_{i=1}^{n}\left(e^{\frac{i}{n}}\right)\cdot\left(\frac{1}{n}\right) = \sum_{i} e^{\xi_i}\cdot\left(\frac{1}{n}\right)$$

$$= \sum_i f(\xi_i)\cdot\left(\frac{1}{n}\right) \quad \text{check}$$

$$= \sum_i f(\xi_i)\left(\frac{i}{n}-\frac{i-1}{n}\right)$$

$$= \sum_i f(\xi_i)(x_i - x_{i-1})$$

$$= \underbrace{\sum_i}_{=\int} \underbrace{f(\xi_i)}_{=f(x)} \underbrace{\Delta_i}_{=dx} \xrightarrow{n\to\infty} \int_a^b f(x)\,dx$$

So, the limit $= \int_0^1 e^x\,dx$

$$\overset{\text{FTC}}{=} \left[e^x\right]_0^1 = \boxed{e-1}. \; ;)$$