

3A03 Tutorial 2

TA: Jeff

email: marshj16@mcmaster.ca

OH: Mon 11:30 AM (HH 403)

Tutorial recordings + notes posted

on course website.

PS Exercise 7 of § 6.1.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show that

$g(x) = |f(x)|$ is differentiable at c

if and only if $f'(c) = 0$.

$$\text{Sol'n: } \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} = \lim_{x \rightarrow c} \frac{|f(x)| - |f(c)|}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{|f(x)|}{x - c}$$

$$\lim_{x \rightarrow c^+} \frac{|f(x)|}{x - c} = \lim_{x \rightarrow c^+} \left| \frac{f(x)}{x - c} \right|$$

$$= \left| \lim_{x \rightarrow c^+} \frac{f(x)}{x - c} \right| = \left| \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \right|$$

$$= |f'(c)|.$$

$$\lim_{x \rightarrow c^-} \frac{|f(x)|}{x - c} = \lim_{x \rightarrow c^-} - \left| \frac{f(x)}{x - c} \right|$$

$$= - \left| \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \right| = -|f'(c)|.$$

So, $\lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c}$ exists

$$\text{iff } \lim_{x \rightarrow c^+} \frac{g(x) - g(c)}{x - c} = \lim_{x \rightarrow c^-} \frac{g(x) - g(c)}{x - c}$$

$$\text{iff } |f'(c)| = -|f'(c)|$$

$$\text{iff } f'(c) = 0.$$



BS § 6.1.

13. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that

$$f'(c) = \lim (n\{f(c + 1/n) - f(c)\}).$$

However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

Sol'n :

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{n \rightarrow \infty} \frac{f(c + \frac{1}{n}) - f(c)}{1/n}$$

= as desired.

Now take $f(x) = |x|$, $c = 0$.

$$\lim_{n \rightarrow \infty} \left[f\left(c + \frac{1}{n}\right) - f(c) \right] n$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot n = 1.$$

BS § 6.2.

7. Use the Mean Value Theorem to prove that $(x-1)/x < \ln x < x-1$ for $x > 1$. [Hint: Use the fact that $D \ln x = 1/x$ for $x > 0$.]

Sol'n: $f(x) = \ln(x)$. $[a, b] = [1, x]$.

f is diff. on (a, b) and cts.

on $[a, b]$.

So: By MVT, $\exists c \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Leftrightarrow \frac{\ln(x) - \ln(1)}{x - 1} = \ln'(c)$$

$$\frac{\ln(x) - 0}{x-1} = \frac{1}{c}$$

$$\ln(x) = \frac{x-1}{c}$$

$$\text{Now: } c \in (a, b) \Rightarrow c \in (1, x)$$

$$\Rightarrow 1 < c < x$$

$$\Rightarrow \frac{1}{x} < \frac{1}{c} < 1$$

$$\Rightarrow \frac{x-1}{x} < \frac{x-1}{c} < x-1$$

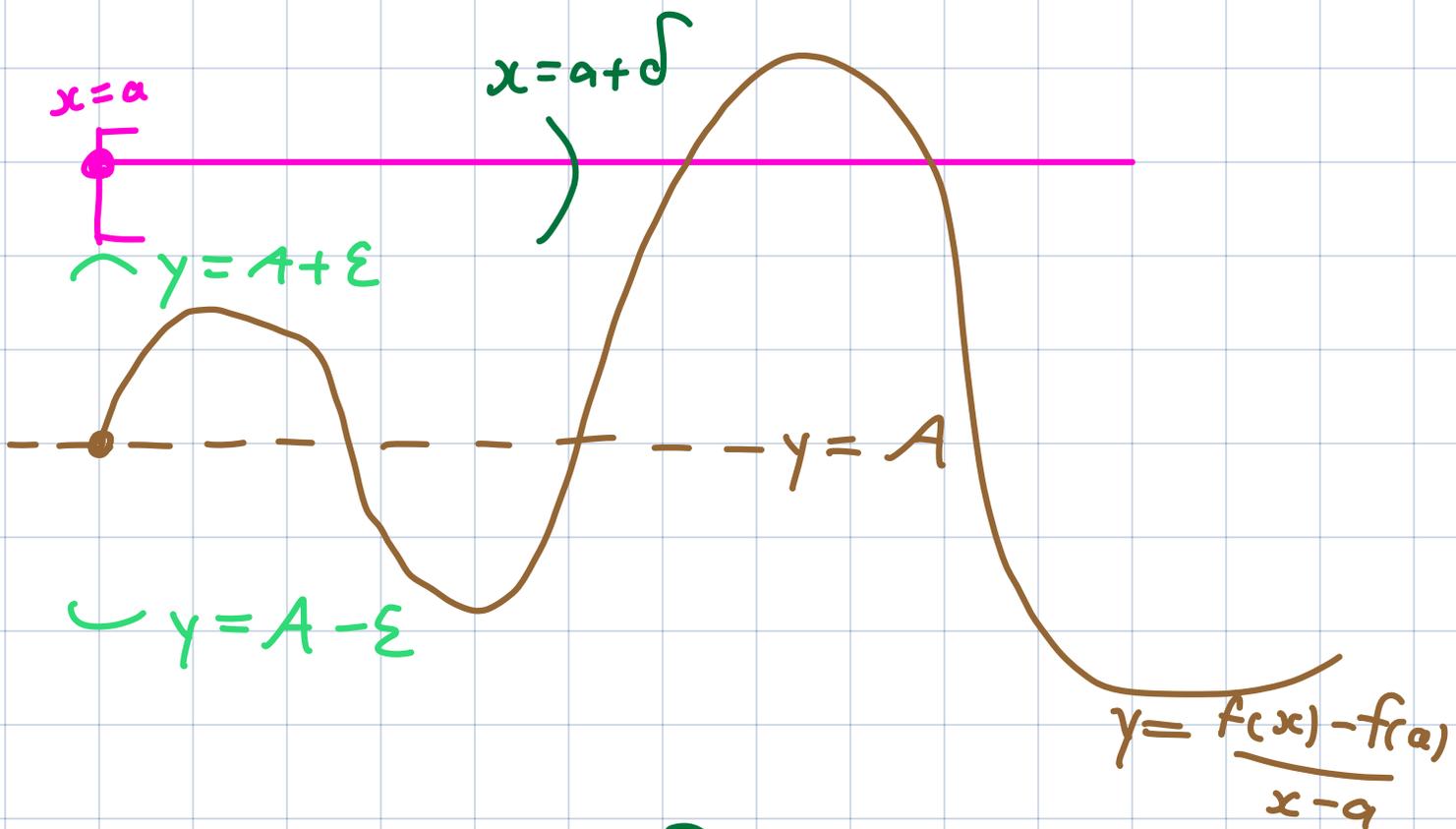
$\underbrace{\hspace{1.5cm}}_{\ln(x)}$

$$\Rightarrow \frac{x-1}{x} < \ln(x) < x-1$$



BS § 6.2.

8. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) . Show that if $\lim_{x \rightarrow a} f'(x) = A$, then $f'(a)$ exists and equals A . [Hint: Use the definition of $f'(a)$ and the Mean Value Theorem.]



Let $\epsilon > 0$. Want $\delta > 0$ s.t.

$$a < x < a + \delta \implies \left| \frac{f(x) - f(a)}{x - a} - A \right| < \epsilon.$$

Let $\delta_0 > 0$ be s.t. $a < x < a + \delta_0$

$$\implies |f'(x) - A| < \epsilon.$$

Now let $x \in (a, a + \delta_0)$ and

apply MVT to f on $[a, x]$.

Then $\exists a < c < x$ s.t.

$$f'(c) = \frac{f(x) - f(a)}{x - a}. \quad \text{But}$$

$$a < c < x \quad \Rightarrow \quad a < c < a + \delta_0$$

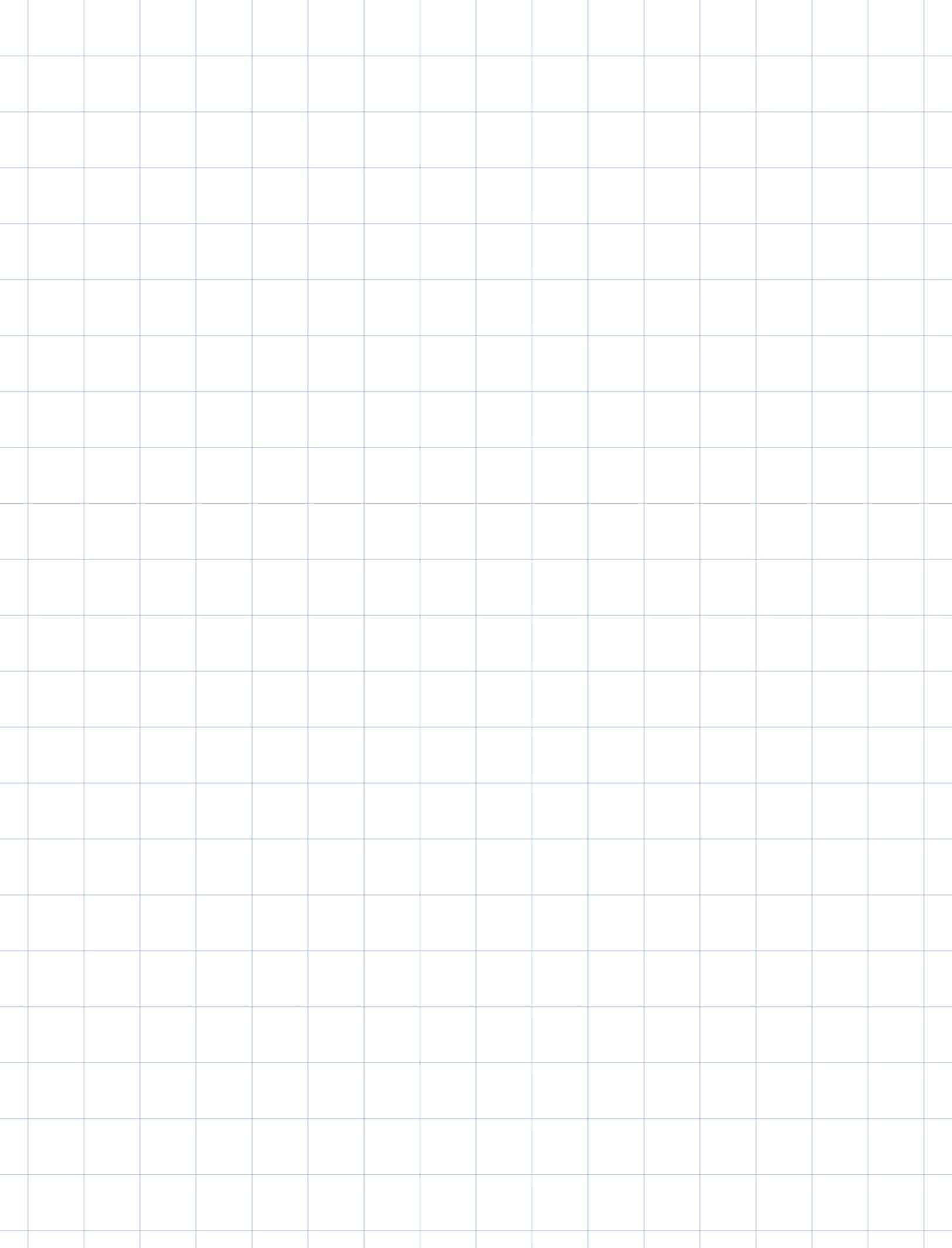


So, $|f'(c) - A| < \varepsilon$ by construction of δ_0 .

$$\therefore \left| \frac{f(x) - f(a)}{x - a} - A \right| < \varepsilon.$$

This holds for any $x \in (a, a + \delta_0)$,

so we are done!



BS Exercise 7 of § 6.1.

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at c and that $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

Sol'n:

$$\lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} = \lim_{x \rightarrow c} \frac{|f(x)| - |f(c)|}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{|f(x)| - |0|}{x - c} = \lim_{x \rightarrow c} \frac{|f(x)|}{x - c}$$

$$\lim_{x \rightarrow c} \text{ exists iff } \lim_{x \rightarrow c^+} = \lim_{x \rightarrow c^-}$$

$$\lim_{x \rightarrow c^+}$$



c x

$$c < x$$

$$|x - c| = x - c$$

$$\lim_{x \rightarrow c^+} \frac{|f(x)|}{x - c} = \lim_{x \rightarrow c^+} \frac{|f(x)|}{|x - c|}$$

$$= \lim_{x \rightarrow c^+} \left| \frac{f(x)}{x - c} \right| \stackrel{?}{=} \left| \lim_{x \rightarrow c^+} \frac{f(x)}{x - c} \right|$$

justified
because

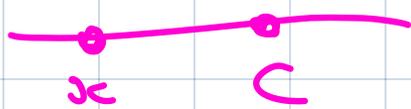
f is cts.

$$= \left| \lim_{x \rightarrow c^+} \frac{f(x) - 0}{x - c} \right|$$

$$= \left| \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \right|$$

$$= |f'(c)|$$

$$\lim_{x \rightarrow c^-}$$



x c

$$- |x - c| = x - c$$

$$x < c$$

$$\lim_{x \rightarrow c^-} \frac{|f(x)|}{x-c} = \lim_{x \rightarrow c^-} \frac{|f(x)|}{-|x-c|}$$

$$= - \lim_{x \rightarrow c^-} \frac{|f(x)|}{|x-c|} = - \lim_{x \rightarrow c^-} \left| \frac{f(x)}{x-c} \right|$$

$$= - \left| \lim_{x \rightarrow c^-} \frac{f(x)}{x-c} \right| = - \left| \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x-c} \right|$$

$$= - \left| \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x-c} \right| = - |f'(c)|$$

lim exists iff

$$\lim_{x \rightarrow c^+} = \lim_{x \rightarrow c^-}$$

$$\parallel \quad \parallel$$
$$|f'(c)| \quad -|f'(c)|$$

So, $g'(c)$ exists iff $|f'(c)| = -|f'(c)|$

iff. $f'(c) = 0$. \therefore)

$$|x-c| = \begin{cases} x-c, & x-c \geq 0 \\ -(x-c), & x-c \leq 0 \end{cases}$$

\uparrow
 $x < c$

$$|x-c| = -(x-c)$$

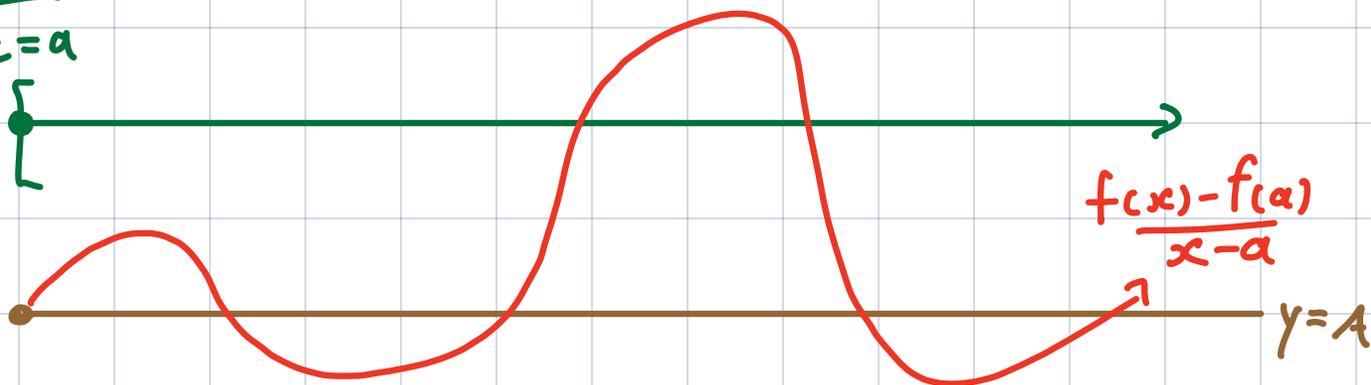
$$\Leftrightarrow -|x-c| = x-c$$

BS §5.2

8. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) . Show that if $\lim_{x \rightarrow a} f'(x) = A$, then $f'(a)$ exists and equals A . [Hint: Use the definition of $f'(a)$ and the Mean Value Theorem.]

Sol'n:

$x=a$



WTS: $\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = A$

ii $\lim_{x \rightarrow a^+} f'(x)$

Proof: Let $\varepsilon > 0$.

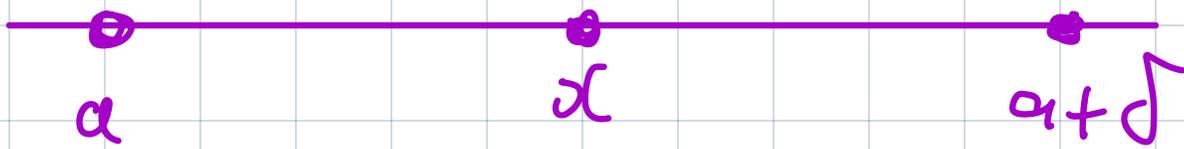
Want: $\delta > 0$ s.t. $a < x < a + \delta$,

then $\left| \frac{f(x) - f(a)}{x - a} - A \right| < \varepsilon$.

Know: $A := \lim_{x \rightarrow a^+} f'(x)$.

So, there exists $\delta > 0$ s.t.

if $a < x < a + \delta$, then $|f'(x) - A| < \varepsilon$.



Let $a < \underline{x} < a + \delta$. Apply MVT to f on $[a, x]$. By the MVT,

there exist $a < c < x$ such

$$\text{that } f'(c) = \frac{f(x) - f(a)}{x - a}.$$



$$a < \underline{c} < a + \delta.$$

By construction of δ ,

$$|f'(c) - A| < \varepsilon.$$



$$\left| \frac{f(x) - f(a)}{x - a} - A \right| < \varepsilon.$$

Since $x \in (a, a + \delta)$ was arbitrary, this is our desired δ !!! QED.
 :))

§6.1

13. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that

$$f'(c) = \lim (n\{f(c + 1/n) - f(c)\}).$$

However, show by example that the existence of the limit of this sequence does not imply the existence of $f'(c)$.

THM: Let $f: \mathbb{R} \rightarrow \mathbb{R}$. Let $c, L \in \mathbb{R}$. Then

$\lim_{x \rightarrow c} f(x) = L$ iff. for any sequence $x_n \in \mathbb{R}$

satisfying (1) $x_n \rightarrow c$ AND

(2) $x_n \neq c \quad \forall n \in \mathbb{N}$,

we have $\lim_{n \rightarrow \infty} f(x_n) = L$.

" In metrizable topological spaces, topological convergence is identical to sequential convergence."

Sol'n: $f'(c) := \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$.

Let $x_n = c + \frac{1}{n}$. THEN

(1) $x_n \rightarrow c$

(2) $x_n \neq c \quad \forall n$.

$$f'(c) := \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

THM $\lim_{n \rightarrow \infty} \frac{f(x_n) - f(c)}{x_n - c}$

$$= \lim_{n \rightarrow \infty} \frac{f(c + \frac{1}{n}) - f(c)}{\cancel{(c + \frac{1}{n})} - \cancel{c}}$$

$$= \lim_{n \rightarrow \infty} n \left[f(c + \frac{1}{n}) - f(c) \right] \quad :)$$

For the 2nd part, take $f(x) = |x|$

and $c = 0$. THEN $f'(c)$ DNE.

$$\text{BUT } \lim_{n \rightarrow \infty} n \left[f\left(c + \frac{1}{n}\right) - f(c) \right]$$

$$= \lim_{n \rightarrow \infty} n \left[\left|0 + \frac{1}{n}\right| - |0| \right]$$

$$= \lim_{n \rightarrow \infty} n \left[\frac{1}{n} \right] = \lim_{n \rightarrow \infty} (1) = 1.$$

So the sequential limit exists!!!

BS § 6.2.

7. Use the Mean Value Theorem to prove that $(x-1)/x < \ln x < x-1$ for $x > 1$. [Hint: Use the fact that $D \ln x = 1/x$ for $x > 0$.]

Look at $f(x) = \ln(x)$, $[a, b] = [1, x]$.

Apply MVT! $\exists c \in (1, x)$ s.t.

$$\ln'(c) = \frac{\ln(x) - \ln(1) = 0}{x-1}$$

$$\frac{1}{c} = \frac{\ln(x)}{x-1}$$



$$1 < c < x$$

$$\frac{1}{x} < \frac{1}{c} < 1$$

$$\frac{1}{x} < \frac{\ln(x)}{x-1} < 1$$

$$\frac{x-1}{x} < \ln(x) < x-1 \quad \square$$