3A03 Tutorial 1

TA: Jeff

email: marshj16@mcmaster.ca

OH (for this week): Thu. 11:30 AM
HH403

Tutorials recordings + notes posted on course website.

TBB 7.2.3: Check the differentiability of each of the functions below at $x_0 = 0$.

(a) $f(x) = x|x|$ (b) $f(x) = x\sin\left(\frac{1}{x}\right)$ $(f(0)=0)$

(c) $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ $(f(0)=0)$

(d) $f(x) = \begin{cases} x^2, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

TBB 7.2.6: A function f has a symmetric derivative at a point if

$$f_s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. Show that $f_s'(x) = f'(x)$ at any point at which the latter exists but that $f_s'(x)$ may exist even when f is not differentiable at x.

TBB : 7.2.5: For what positive values of p is the function $f(x) = |x|^p$ differentiable at 0?

TBB 7.2.22: Show that a function f that satisfies an inequality of the form

$$|f(x) - f(y)| \le M\sqrt{|x-y|}$$

for some constant M and all x, y must be everywhere continuous but need not be everywhere differentiable.

Soln to 7.2.3 : (a) $f(x) = x|x|$.

Left derivative?

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0^-} \frac{x|x|-(0)|0|}{x-0}$$

$$= \lim_{x\to 0^-} \frac{x(-x)}{x} = -\lim_{x\to 0^-} (x) = 0.$$

Right derivative?

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x|x|-(0)|0|}{x-0}$$

$$= \lim_{x\to 0^+} \frac{x(x)}{x} = \lim_{x\to 0^+} x = 0.$$

Left derivative = right derivative,

so f is diff. at $x_0 = 0$ w/

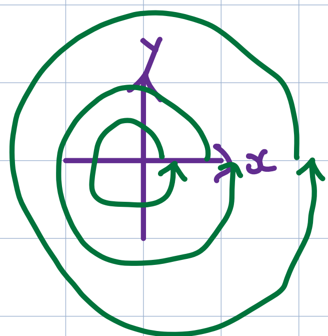
$f'(0) = 0.$

(b) $f(x) = x \sin\left(\frac{1}{x}\right) \qquad (f(0) = 0)$

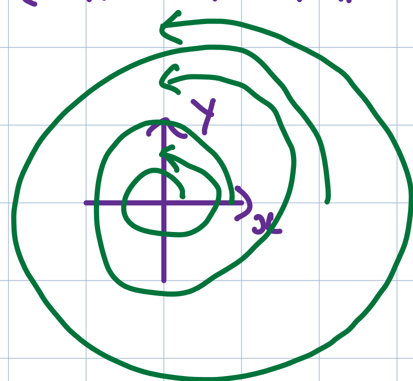
Take $x_n = \frac{1}{2n\pi}$, $\quad y_n = \frac{2}{(4n+1)\pi}$

$x_n = \left(\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \dots\right) \qquad y_n = \left(\frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi}, \dots\right)$

$\frac{1}{x_n}$



$\frac{1}{x_n}$



Evaluate $f'(0)$ with $x_n \longrightarrow 0.$

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{x_n \sin\left(\frac{1}{x_n}\right) - 0}{x_n}$$

$$= \frac{\cancel{\frac{1}{2n\pi}} \sin\left(\frac{1}{1/2n\pi}\right)}{\cancel{1/2n\pi}} = \sin(2n\pi) = 0$$

$$\xrightarrow{n \to \infty} 0.$$

Evaluate $f'(0)$ with y_n.

$$\frac{f(y_n) - f(0)}{y_n - 0} = \frac{\frac{2}{\cancel{(4n+1)\pi}} \sin\left(\frac{1}{2/(4n+1)\pi}\right) - 0}{2/\cancel{(4n+1)\pi}}$$

$$= \sin\left(\frac{4n+1}{2} \cdot \pi\right) = 1$$

$$\xrightarrow{n \to \infty} 1.$$

$$1 \neq 0.$$

x_n-derivative $\neq$ y_n-derivative.

So, f <u>not</u> diff. at $x_0 = 0$.

(c) $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ $(f(0) = 0)$

Trapping principle.

$$|f(x)| = \left|x^2 \sin\left(\frac{1}{x}\right)\right|$$
$$= |x^2| \left|\sin\left(\frac{1}{x}\right)\right|$$
$$\leq |x^2| (1) = x^2,$$

so f is diff. at $x_0 = 0$ w/

$f'(0) = 0$ by the trapping principle.

(d)
$$f(x) = \begin{cases} x^2, & x \text{ rational} \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

$$= \frac{1}{x}\begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} \frac{x^2}{x}, & x \in \mathbb{Q} \\ \frac{0}{x}, & \text{else} \end{cases}$$

$$= \begin{cases} x, & x \in \mathbb{Q} \\ 0, & \text{else} \end{cases}$$

$$\xrightarrow{x \to 0} \begin{cases} 0, & x \in \mathbb{Q} \\ 0, & \text{else} \end{cases} = 0$$

$$\Rightarrow f'(0) = 0.$$

<u>TBB 7.2.6</u>: Assume $f'(x)$ exists.

Then $\frac{f(x+h) - f(x-h)}{2h}$

$$= \frac{1}{2}\left(\frac{f(x+h) - f(x-h)}{h}\right)$$

$$= \frac{1}{2}\left(\frac{f(x+h) - f(x)}{h} - \frac{f(x-h) - f(x)}{h}\right)$$

$$\xrightarrow{h \to 0} f'(x).$$

Example where $f_s'(x)$ exists but $f'(x)$ does not: $f(x) = |x|$.

$f'(0)$ DNE.

$$f_s'(0) : \lim_{h \to 0} \frac{f(0+h) - f(0-h)}{2h}$$

$$= \frac{|h| - |-h|}{2h} = \frac{|h| - |h|}{2h} = \frac{0}{2h} = 0$$

$$\xrightarrow{h \to 0} 0 \Rightarrow f_s'(0) = 0.$$

TBB 7.2.5: $f(x) = |x|^p$

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|^p}{h}$$

$$= \lim_{h\to 0} \begin{cases} h^{p-1}, & h>0 \\ -h^{p-1}, & h<0 \end{cases}$$

CI: $p > 1$.

Then $h^{p-1} \xrightarrow{h\to 0} 0 \quad (h>0)$

$-h^{p-1} \xrightarrow[h\to 0]{} 0 \quad (h<0)$

$\Rightarrow f'(0) = 0.$

CII: $p=1$

$h^{p-1} = h^0 = 1 \xrightarrow{h\to 0} 1 \quad (h>0)$

$-h^{p-1} = -h^0 = -1 \xrightarrow{h\to 0} -1 \quad (h<0)$

$\Rightarrow f'(0)$ DNE.

CIII : $p<1$

$h^{p-1} \longrightarrow \infty \quad (h>0)$

$h^{p-1} \longrightarrow -\infty \quad (h<0)$

$\Rightarrow f'(0)$ DNE.

TBB 7.2.22: Continuity:

WTS: $\lim_{x\to a} f(x) = f(a)$

$\Longleftrightarrow$

$\lim_{x\to a} |f(x)-f(a)| = 0.$

$$\lim_{x\to a} |f(x)-f(a)| \le \lim_{x\to a} M\sqrt{|x-a|}$$

$$\overset{\uparrow}{=} M\sqrt{\lim_{x\to a}|x-a|} = M\cdot 0 = 0.$$

continuity of $\sqrt{\ }$

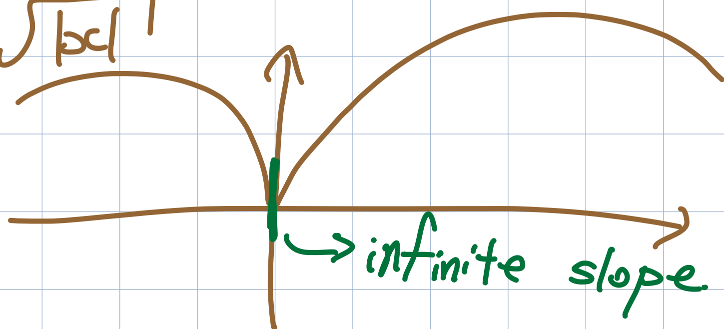
:)

Example of f w/ $|f(x)-f(y)|$

$$\le M\sqrt{|x-y|}$$

but f <u>not</u> everywhere diff.

$f(x) = \sqrt{|x|}$



So <u>not</u> diff. at $x=0$.

<u>Proof of inequality</u>: $|\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|}$

<u>CI</u>: $y=0$. easy.

$\underline{C\,\mathrm{II}}$: $y \neq 0$.

$$x - 2\sqrt{xy} + y \leq |x-y|$$

WLOG, assume $x \geq y$

$$\Leftrightarrow$$

$$\cancel{x} - 2\sqrt{xy} + y \leq \cancel{x} - y$$

$$\Leftrightarrow$$

$$2y \leq 2\sqrt{xy}$$

$$\Leftrightarrow$$

$$y \leq \sqrt{xy}$$

$$\Leftrightarrow$$

$$y^2 \leq xy$$

$$\Leftrightarrow$$

$$y \cdot \cancel{y} \leq x \cdot \cancel{y}$$

$$\Leftrightarrow$$

$$y \leq x.$$

✓✓✓

TBB 7.2.3: Check the differentiability of each of the functions below at $x_0 = 0$.

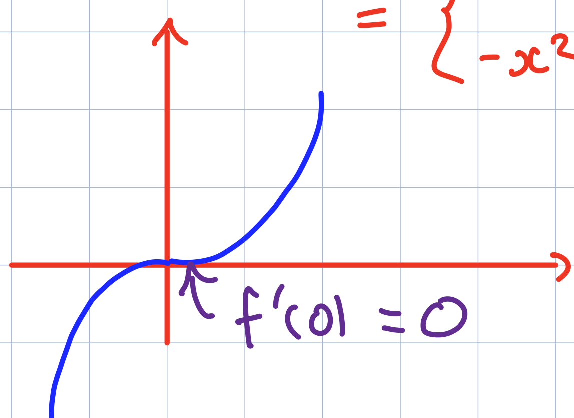
(a) $f(x) = x|x|$ (b) $f(x) = x\sin\left(\frac{1}{x}\right)$ $(f(0)=0)$

(c) $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ $(f(0)=0)$

(d) $f(x) = \begin{cases} x^2, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

Sol'n: (a) $f(x) = x|x|$

$$= \begin{cases} x^2, & x > 0 \\ -x^2, & x \leq 0 \end{cases}$$



$f'(0) = 0$

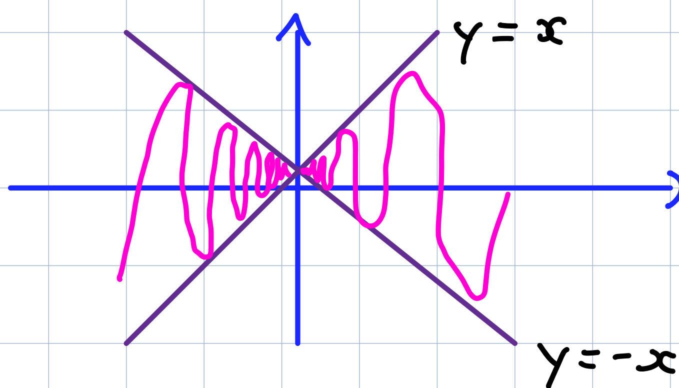
$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x|x| - (0)|0|}{x}$$

$$= \lim_{x \to 0} \frac{\cancel{x}|x|}{\cancel{x}} = \lim_{x \to 0} |x| = 0.$$

So, $f'(0) = 0.$

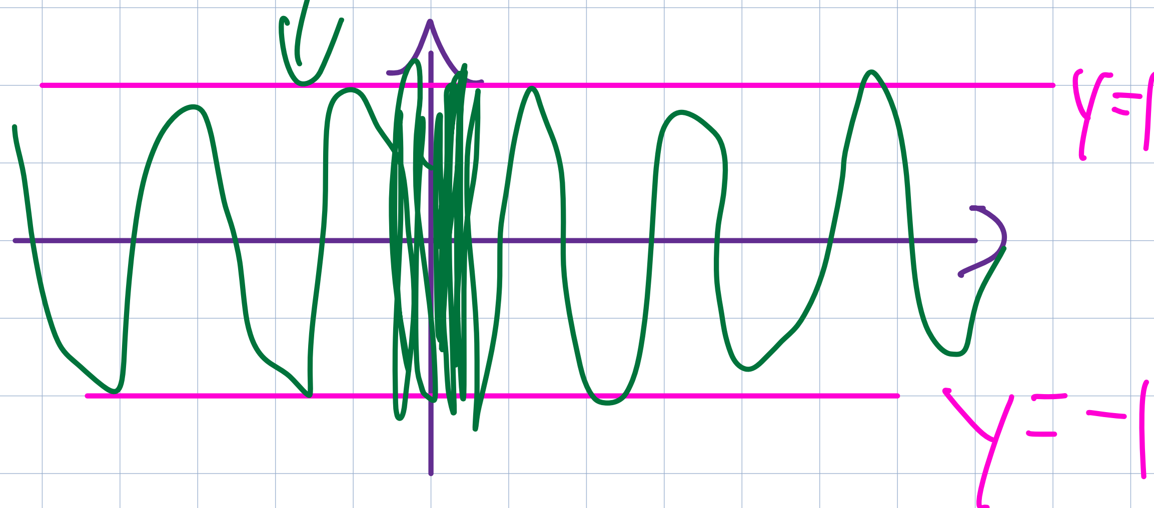
(b) $f(x) = x \sin\left(\frac{1}{x}\right) \quad (f(0) = 0)$



$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

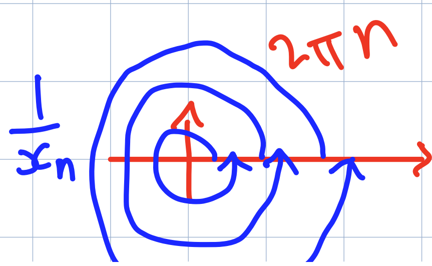
$$= \lim_{x \to 0} \frac{\cancel{x} \sin\left(\frac{1}{x}\right) - 0}{\cancel{x}}$$

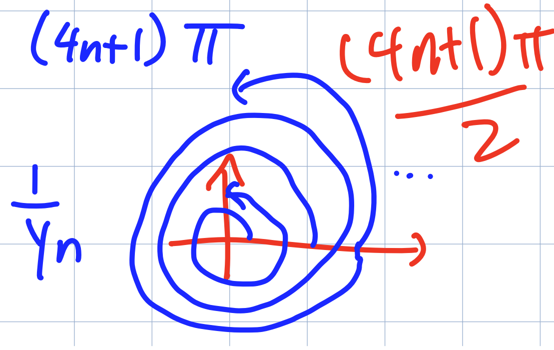
$$= \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$



2 sequences $x_n, y_n \longrightarrow 0$. $(n>0)$

$x_n = \frac{1}{2\pi n}$, $y_n = \frac{2}{(4n+1)\pi}$





$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = \lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right)$$

if it exists

$$= \lim_{n\to\infty} \sin\left(\frac{1}{1/2\pi n}\right) = \lim_{n\to\infty} \sin(2\pi n)$$

$$= \lim_{n\to\infty} [0] = 0.$$

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = \lim_{n\to\infty} \sin\left(\frac{1}{y_n}\right)$$

if it exists

$$= \lim_{n\to\infty} \sin\left(\frac{2}{1/(4n+1)\pi}\right)$$

$$= \lim_{n\to\infty} \sin\left(\frac{(4n+1)\pi}{2}\right)$$

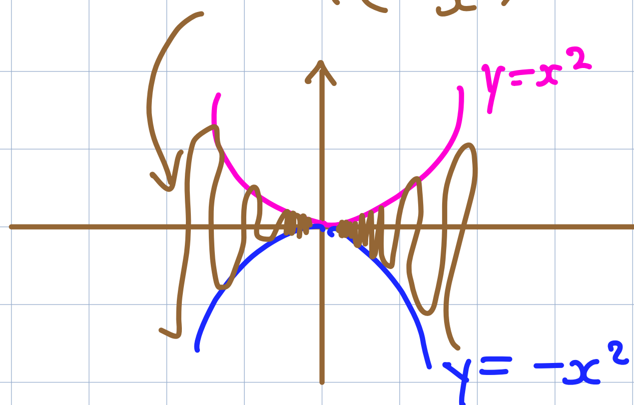
$$= \lim_{n\to\infty} (1) = 1.$$

So... if $f'(0)$ exists, then

$0 = 1$.

So, $f'(0)$ DNE. :)

(c) $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ $\quad (f(0) = 0)$



Follows by the "Trapping principle".

$$|f'(0)| = \left|\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0}\left|\frac{x^2\sin\left(\frac{1}{x}\right) - 0}{x - 0}\right|$$

$$= \lim_{x\to 0} |x| \underbrace{\left|\sin\left(\frac{1}{x}\right)\right|}_{\leq 1} \leq \lim_{x\to 0} |x| = 0.$$

:)

So, $f'(0) = 0$ by Squeeze THM.

<u>TBB 7.2.6</u>: A function f has a <u>symmetric derivative</u> at a point if

$$f_s'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. Show that $f_s'(x) = f'(x)$ at any point at which the latter exists but that $f_s'(x)$ may exist even when f is not differentiable at x.

Sol'n: Assume $f'(x)$ exists.

WTS: $f_s'(x) = f'(x)$.

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

$$= \lim_{h\to 0} \frac{1}{2}\left[\frac{f(x+h) - f(x-h)}{h}\right]$$

$$= \lim_{h\to 0} \frac{1}{2}\left[\frac{f(x+h) - f(x) \overbrace{\ }^{+0} + f(x) - f(x-h)}{h}\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] + \frac{1}{2}\left[\lim_{h\to 0} \frac{f(x) - f(x-h)}{h}\right]$$

this step is justified only if both limits exists. We will see that both limits exist.

$$= \frac{1}{2}\left[f'(x)\right] + \frac{1}{2}\left[\lim_{h\to 0} \frac{f(x)-f(x-h)}{h}\right]$$

$$= \frac{1}{2} f'(x) - \frac{1}{2}\left[\lim_{h\to 0} \frac{f(x-h)-f(x)}{h}\right]$$

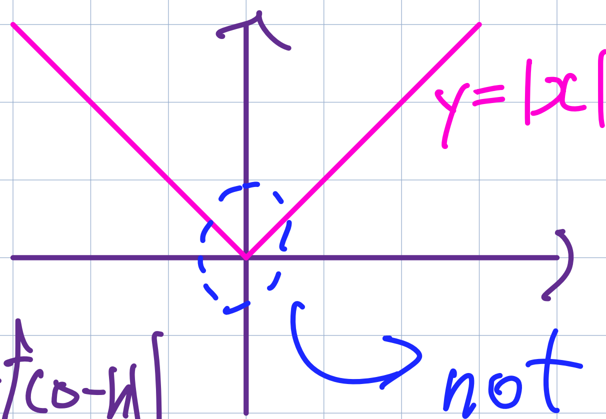
$$= \frac{1}{2} f'(x) - \frac{1}{2}\left[\lim_{-h\to 0} \frac{f(x-(-h))-f(x)}{(-h)}\right]$$

$$= \frac{1}{2} f'(x) - \frac{1}{2}\underbrace{\left[-\lim_{+h\to 0} \frac{f(x+h)-f(x)}{h}\right]}_{=f'(x)}$$

$$= \frac{1}{2} f'(x) + \frac{1}{2} f'(x)$$

$$= f'(x)$$:) (some abuse of notation)

Part 2: find f and x s.t. $f'(x)$ DNE but $f'_s(x)$ does.



not diff at $x=0$

$f(x) = |x|.$

$$f'_s(0) = \lim_{h\to 0} \frac{f(0+h) - f(0-h)}{2h}$$

$$= \lim_{h\to 0} \frac{|0+h| - |0-h|}{2h}$$

$$= \lim_{h\to 0} \frac{|h| - \overbrace{|-h|}^{|h|}}{2h}$$

$$= \lim_{h\to 0} \frac{|h| - |h|}{2h} = \lim_{h\to 0} \frac{0}{2h}$$

$= \lim_{h \to 0} 0 = 0$:)

So $f_s'(0)$ exists!!!

Yay!

Given $f: \mathbb{R} \longrightarrow \mathbb{R}$,

$\lim_{x \to c} f(x) = L$ iff.

for any sequence of real numbers x_n converging to c, $\left[\begin{array}{l} x_n \longrightarrow c \\ \lim_{n \to \infty} x_n = c \end{array} \right]$
the sequence $f(x_n)$ converges to L. $\left[\begin{array}{l} f(x_n) \longrightarrow L \\ \lim_{n \to \infty} f(x_n) = L \end{array} \right]$

$g(x) = \sin\left(\frac{1}{x}\right)$

$g(0) = 0$

We found 2 sequences x_n, y_n both converging to 0.

But, $g(x_n) \longrightarrow 0$

$g(y_n) \longrightarrow 1.$