Def. A <u>partition</u> of $[a,b]$ is a finite collection (?) of points $\{t_0, \dots, t_n\} \subseteq [a,b]$ with $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$.

Def. If $f: [a,b] \to \mathbb{R}$ is bounded, define

$$m_i := \inf\{f(x) : x \in [t_{i-1}, t_i]\}$$
$$M_i := \sup\{f(x) : x \in [t_{i-1}, t_i]\}$$

The <u>lower sum</u> (resp. upper) of f for P is

$$L(f,P) := \sum_{i=1}^{n} m_i (t_i - t_{i-1})$$

$$\left(\text{resp. } U(f,P) := \sum_{i=1}^{n} M_i (t_i - t_{i-1})\right).$$

Define
$$L(f) := \sup\{L(f,P)\}$$
$$U(f) := \inf\{U(f,P)\}$$

Facts: (1) For any partition P of $[a,b]$, $L(f,P) \leq U(f,P)$

(2) If P, Q are partitions with $P \subseteq Q$, then
$$L(f,P) \leq L(f,Q)$$
$$U(f,Q) \leq U(f,P)$$

(3) For any two partitions P_1 and P_2,
$L(f,P_1) \leq U(f,P_2)$ (generalizes (1)).

(4) $L(f) \leq U(f)$.

Big idea: When $L(f) = U(f)$, we can define the shared value to be the integral (i.e. the area under the curve).

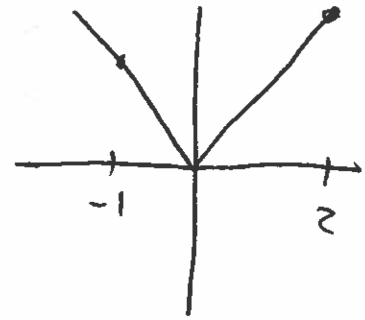
BS 7.4.1 Let $f(x) = |x|$ for $-1 \leq x \leq 2$. 5

Calculate $L(f,P)$ and $U(f,P)$ for the partitions

a) $P_1 = \{-1, 0, 1, 2\}$

b) $P_2 = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$

Sol. a) Here, $t_0 = -1$, $t_1 = 0$, $t_2 = 1$, $t_3 = 2$.



Recall, $m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}$

$M_i = \sup\{$ " " $\}$

We have
$m_1 = 0 \qquad M_1 = 1$
$m_2 = 0 \qquad M_2 = 1$
$m_3 = 1 \qquad M_3 = 2$

So $L(f,P_1) = \sum_{i=1}^{3} m_i(t_i - t_{i-1}) = \sum_{i=1}^{3} m_i(1) = 1$

$U(f,P_1) = \sum_{i=1}^{3} M_i(t_i - t_{i-1}) = \sum_{i=1}^{3} M_i(1) = 4.$

b) Now, we have a finer partition, so we expect $L(f,P_2)$ and $U(f,P_2)$ to be closer to the "actual area under the curve"

We have
$m_1 = \frac{1}{2} \qquad M_1 = 1$
$m_2 = 0 \qquad M_2 = \frac{1}{2}$
$m_3 = 0 \qquad M_3 = \frac{1}{2}$
$m_4 = \frac{1}{2} \qquad M_4 = 1$
$m_5 = 1 \qquad M_5 = \frac{3}{2}$
$m_6 = \frac{3}{2} \qquad M_6 = 2$

So $L(f,P_2) = \sum_{i=1} m_i(\frac{1}{2}) = \frac{1}{2}(\frac{1}{2}+0+0+\frac{1}{2}+1+\frac{3}{2})$.
$= \frac{7}{4} = 1.75$

4

$U(f,P_2) = \sum_{i=1}^{n} M_i(\frac{1}{2}) = \frac{1}{2}(1+\frac{1}{2}+\frac{1}{2}+1+\frac{3}{2}+2)$
$= \frac{13}{4} = 3.25$

Notice the lower sum increased and the upper one decreased. The actual area under the curve is 2.5, so it appears the sums are getting closer to 2.5 as the partitions get finer. ∎

<u>BS 7.~~4~~.3</u> Let f and g be bounded on $I=[a,b]$.
Suppose $f(x) \le g(x)$ $\forall x \in I$.
Show $L(f) \le L(g)$ and $U(f) \le U(g)$.

<u>Sol</u>. We'll do the first inequality (other one similar).
We want to show that
$$\sup\{L(f,P) : \cancel{P \text{ partition}}\} \le \sup\{L(g,P)\}$$

By definition of the sup as 'LUB', it suffices to show that $L(f,Q) \le \sup\{L(g,P)\}$ for any partition Q (i.e. $\sup\{L(g,P)\}$ is an upper bound for the set of lower sums of f.

Notice that if $L(f,Q) \le L(g,Q)$, then since $\sup\{L(g,P)\}$ is an upper bound for $L(g,Q)$, we would get $L(f,Q) \le L(g,Q) \le \sup\{L(g,P)\}$ which is exactly what we want to show.

So, let's fix Q a partition and show (5)
$L(f,Q) \le L(g, Q)$. We have
$L(f,Q) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$ where ~~$m_i = \inf\{f(x):$~~
$m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}$

Since $f(x) \le g(x)$ ~~and m_i is~~ $\forall x \in [t_{i-1}, t_i]$
and m_i is a lower bound for $f(x)$ on $[t_{i-1}, t_i]$,
it's also a lower bound for $g(x)$ on $[t_{i-1}, t_i]$.
But ~~iff~~ if we define $n_i = \inf\{g(x) : x \in [t_{i-1}, t_i]\}$
then n_i is the GLB for $g(x)$ on $[t_{i-1}, t_i]$.
So $m_i \le n_i$ for each i.

Hence $L(f,Q) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \le \sum_{i=1}^{n} n_i(t_i - t_{i-1}) = L(g, Q)$
which we wanted to show, so we're done. ∎

<u>BS 7.4.6</u> Let $f : [0,2] \to \mathbb{R}$ be given by $f(x) = 1$ for $x \ne 1$, $f(1) = 0$. Show that $L(f) = U(f)$. What is the shared value?

<u>Sol</u>. Start by finding $L(f) = \sup\{L(f,P)\}$.
Let P be an arbitrary partition. Try to bound $L(f,P)$:
$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$. Can we bound m_i?

For all $x \in [t_{i-1}, t_i]$, $f(x) \le 1$, so 1 is an upper bound for f on $[t_{i-1}, t_i]$. But m_i is a lower bound for f on $[t_{i-1}, t_i]$, so certainly $m_i \le 1$.
So $L(f,P) \le \sum_{i=1}^{n} (1)(t_i - t_{i-1}) = 2$.
Hence $\sup\{L(f,P)\} \le 2$.

We claim $\sup\{\cancel{\text{all}}\, L(f,P)\} = 2$.
Since 2 is an upper bound for $\{L(f,P) : P \text{ partition}\}$, it suffices to show that $\forall \varepsilon > 0$, $\exists$ a partition P such that $2-\varepsilon \le L(f,P) \le 2$.

So let $\varepsilon > 0$ be given. Choose P to be the partition $\{0, 1-\frac{\varepsilon}{4}, 1+\frac{\varepsilon}{4}, 2\}$. Observe that

$$
\begin{aligned}
L(f,P) &= (1)(1-\tfrac{\varepsilon}{4}) + (0)((1+\tfrac{\varepsilon}{4})-(1-\tfrac{\varepsilon}{4})) + (1)(2-(1+\tfrac{\varepsilon}{4})) \\
&= 1-\tfrac{\varepsilon}{4} + (1-\tfrac{\varepsilon}{4}) \\
&= 2-\tfrac{\varepsilon}{2}
\end{aligned}
$$

so $2-\varepsilon < \cancel{\text{all}}\, L(f,P) \le 2$.
So we conclude that $L(f) = \sup\{L(f,P)\} = 2$.

What about $U(f)$? We already know $2 = L(f) \le U(f)$. So $2 \le \inf\{U(f,P)\}$.

<u>Q</u>. Is there a partition P such that $U(f,P) = 2$?
In that case, we have $\inf\{U(f,P)\} \le 2$ as desired.

<u>A</u>. Yes! Choose $P = \{0, 2\}$. Then

$$
\begin{aligned}
U(f,P) &= \inf\{f(x) : x \in [0,2]\} \cdot (2-0) \\
&= 1 \cdot 2 = 2.
\end{aligned}
$$

So $U(f) = \inf\{U(f,P)\} = 2$ and $L(f) = 2 = U(f)$. ∎

BS 7.4.8 Let f be continuous on $I = [a,b]$ and assume $f(x) \geq 0$ $\forall x \in I$. Prove that if $L(f)=0$, then $f(x)=0$ for all $x \in I$. (7)

Sol. Suppose towards a contradiction that $f(c) > 0$ for some $c \in I$. Let's assume c is an interior point of I. Since f is continuous, there exists some neighborhood $(c-\delta, c+\delta)$ of c such $\delta > 0$ that $f(x) > \frac{f(c)}{2}$ for all $x \in (c-\delta, c+\delta)$. ~~In fact, you can assume f(x)~~

Consider the partition $P = \{a, c-\delta, c+\delta, b\}$.

We have
$$\begin{aligned} L(f,P) &= m_1(c-\delta-a) + m_2(c+\delta-(c-\delta)) + m_3(b-c-\delta) \\ &\geq 0(c-\delta-a) + \frac{f(c)}{2}(2\delta) + 0(b-c-\delta) \\ &= f(c)\cdot\delta \\ &> 0 \end{aligned}$$

But since $L(f) = \sup\{L(f,P)\}$, ~~we have~~ $L(f)$ ~~$0 = L(f)$~~ is an upper bound for $L(f,P)$ so $0 < L(f,P) \leq L(f) = 0$, a contradiction.

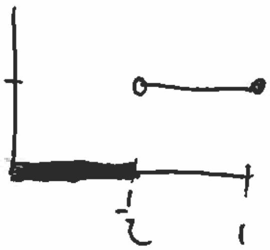
Thus $f(x)=0$ $\forall x \in I$.

BS Ex 7.4.7d) Let $f:[0,1]\to I$ be given by
$$f(x)=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x \in \mathbb{R}\setminus\mathbb{Q}\end{cases}$$
Find $L(f)$ and $U(f)$.

Sol. Let $P=\{t_0,\dots,t_n\}$ be any partition.
Since any interval $[t_{i-1},t_i]$ contains irrational and rational numbers (density), we have
$$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1})=0$$
$$U(f,P)=\sum_{i=1}^{n} M_i(t_i-t_{i-1})=\sum_{i=1}^{n}(1)(t_i-t_{i-1})=(1-0)=1.$$
So $L(f)=\sup\{0\}=0$.
$U(f)=\inf\{1\}=1$.

Note that we always have $L(f)\le U(f)$, but in this case we have strict inequality.

<u>BSB 7.5.7</u> a) Let $g(x) = \begin{cases} 0 , & 0 \le x \le \frac{1}{2} \\ 1 , & \frac{1}{2} < x \le 1 \end{cases}$



Show that g is integrable with $\int_0^1 g = \frac{1}{2}$.

b) Does the conclusion hold if we instead define $g(\frac{1}{2}) = \cancel{0}\ 1$?

<u>Sol.</u> a) Let $\varepsilon > 0$ be given. Define $P_\varepsilon = \{0, \frac{1}{2} - \frac{\varepsilon}{6}, \frac{1}{2} + \frac{\varepsilon}{6}, 1\}$.
If we can show that $U(g, P_\varepsilon) - L(g, P_\varepsilon) < \varepsilon$, then we can conclude that g is integrable.

$$U(g, P_\varepsilon) = 0\left(\tfrac{1}{2} - \tfrac{\varepsilon}{6}\right) + 1\left(\tfrac{\varepsilon}{3}\right) + 1\left(1 - \left(\tfrac{1}{2} + \tfrac{\varepsilon}{6}\right)\right)$$
$$= \frac{\varepsilon}{3} + \frac{1}{2} - \frac{\varepsilon}{6}$$
$$= \frac{1}{2} + \frac{\varepsilon}{6}$$

$$L(g, P_\varepsilon) = 0\left(\tfrac{1}{2} - \tfrac{\varepsilon}{6}\right) + 0\left(\tfrac{\varepsilon}{3}\right) + 1\left(1 - \left(\tfrac{1}{2} + \tfrac{\varepsilon}{6}\right)\right)$$
$$= \frac{1}{2} - \frac{\varepsilon}{6}$$

So $U(g, P_\varepsilon) - L(g, P_\varepsilon) = \left(\frac{1}{2} + \frac{\varepsilon}{6}\right) - \left(\frac{1}{2} - \frac{\varepsilon}{6}\right) = \frac{\varepsilon}{3} < \varepsilon$
so g is integrable. What is $\int_0^1 g$?

We must have $\inf_{P'} \{U(g, P')\} \le U(g, P_\varepsilon)\ \forall \varepsilon > 0$
so $\inf_{P'} \{U(g, P')\} \le \frac{1}{2}$. Likewise,
$\sup_{P'} \{L(g, P')\} \ge L(g, P_\varepsilon)\ \forall \varepsilon > 0$ so $\sup_{P'} \{L(g, P')\} \ge \frac{1}{2}$
But then $\frac{1}{2} \le \sup_{P'} \{L(g, P')\} \le \inf_{P'} \{U(g, P')\} \le \frac{1}{2}$

So $\int_0^1 g = \frac{1}{2}$.

b) If $g(\frac{1}{2}) = \frac{1}{3}$, then for the same partition P_ε, we have

$$U(g, P_\varepsilon) = 0\left(\frac{1}{2} - \frac{\varepsilon}{6}\right) + 13\left(\frac{\varepsilon}{3}\right) + 1\left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{6}\right)\right)$$
$$= \frac{13}{3}\varepsilon + \frac{1}{2} - \frac{\varepsilon}{6}$$
$$= \frac{25}{6}\varepsilon + \frac{1}{2}$$

$$L(g, P_\varepsilon) = \frac{1}{2} - \frac{\varepsilon}{6}$$

Again, note that $\inf_{P'}\{U(g, P')\} \leq \frac{1}{2}$ and $\sup_{P'}\{L(g, P')\} \geq \frac{1}{2}$ so

$$\frac{1}{2} \leq \sup_{P'}\{L(g, P')\} \leq \inf_{P'}\{U(g, P')\} \leq \frac{1}{2}$$

so $\sup_{P'}\{L(g, P')\} = \frac{1}{2} = \inf_{P'}\{U(g, P')\}$.

Thus, g is still integrable and $\int_0^1 g = \frac{1}{2}$.

A8 Q1 Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Show f is integrable on any closed subinterval of $[a,b]$.

Sol. A closed subinterval of $[a,b]$ could look like $[a,c]$, $[d,b]$ or $[c,d]$ where $a < c < d < b$. So let's show f is integrable on each of these subintervals.

We will use the ε-P criterion. Let $\varepsilon > 0$. Since f is integrable on $[a,b]$, there exists a partition P_ε of $[a,b]$ s.t.

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Let $Q_\varepsilon = P_\varepsilon \cup \{c, d\}$. This is a refinement of P_ε, so $U(f, Q_\varepsilon) \leq U(f, P_\varepsilon)$ ad

$$L(f, Q_\varepsilon) \geq L(f, P_\varepsilon).$$

Hence $U(f,Q_\varepsilon) - L(f,Q_\varepsilon) < U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$.

This is basically what we want, except Q_ε is a partition of $[a,b]$, not the subintervals. So break Q_ε into three parts:

Define $Q_1 = Q_\varepsilon \cap [a,c]$ so $Q = Q_1 \cup Q_2 \cup Q_3$ is a disjoint union

$Q_2 = Q_\varepsilon \cap [c,d]$

$Q_3 = Q_\varepsilon \cap [d,b]$

It follows that

$$U(f, Q_\varepsilon) = U(f, Q_1) + U(f, Q_2) + U(f, Q_3)$$
$$L(f, Q_\varepsilon) = L(f, Q_1) + L(f, Q_2) + L(f, Q_3)$$

Subtracting these two equations gives

$$\begin{array}{l} [U(f,Q_1) - L(f,Q_1)] \\ \quad + \\ U(f,Q_2) - L(f,Q_2) \\ \quad + \\ U(f,Q_3) - L(f,Q_3) \end{array} = U(f,Q_\varepsilon) - L(f,Q_\varepsilon) < \varepsilon$$

But each of the differences on the right is ≥ 0. So each must be $< \varepsilon$.

Hence f is integrable on each subinterval, as we've found partitions ~~that~~ Q_1, Q_2, Q_3 that satisfy the ε-P criterion

A2 Q2b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Show $\inf\{U(f,P)\} = \frac{1}{2}$, where P is a partition of $[0,1]$.

Sol. Why should $\inf = \frac{1}{2}$? Morally, taking finer partitions should give upper sums closer to the value of $\inf\{U(f,P)\}$.

For each n, consider the partition P_n given by dividing $[0,1]$ into n ~~se~~ equal subintervals:
$P_n = \{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n}{n}\}$. Here $t_i = \frac{i}{n}$ for each $i = 0, \dots, n$.

What is the sup of f on each subinterval $[t_{i-1}, t_i]$? f is increasing, so it occurs at right endpoint (i.e. sup is $f(t_i) = t_i$).

$$\begin{aligned} \text{So } U(f,P_n) &= \sum_{i=1}^{n} t_i (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \frac{i}{n}\left(\frac{1}{n}\right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \end{aligned}$$

Since $\inf\{U(f,P)\} \le \frac{1}{2} + \frac{1}{2n}$ for all n,
$\inf\{U(f,P)\} \le \frac{1}{2}$.

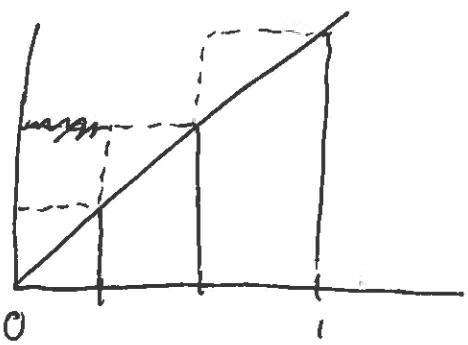
We claim $\inf\{U(f,P)\} = \frac{1}{2}$, so let's show $\inf\{U(f,P)\} \ge \frac{1}{2}$. It suffices to show that ~~all~~ any upper sums are ~~be~~ $\ge \frac{1}{2}$.

So let $P = \{t_0, \ldots, t_n\}$ be a partition of $[0,1]$. We want to bound $U(f,P)$ below by $\frac{1}{2}$.

$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$. What is sup on each subinterval?

By density of rationals in $\mathbb{R}$, can show that $M_i = t_i$ for all i (exercise). So

$U(f,P) = \sum_{i=1}^{n} t_i(t_i - t_{i-1})$. What is this saying geometrically?



$t_i(t_i - t_{i-1})$ is the area of each rectangle.

Notice that this area appears larger than the area of the trapezoid below it, with area

$A_i = (t_i - t_{i-1})t_{i-1} + \frac{1}{2}(t_i - t_{i-1})^2$

So <u>claim</u>: $t_i(t_i - t_{i-1}) \geq A_i$ for each i

<u>Pf</u>. $t_i(t_i - t_{i-1}) = [t_i - t_{i-1} + t_{i-1}](t_i - t_{i-1})$
$= (t_i - t_{i-1})^2 + (t_i - t_{i-1})t_{i-1}$
$\geq (t_i - t_{i-1})^2 + \frac{1}{2}(t_i - t_{i-1})t_{i-1} = A_i$

Why is this useful? Observe that

$A_i = \cancel{t_i t_{i-1}} - t_{i-1}^2 + \frac{1}{2}(t_i^2 - \cancel{2t_i t_{i-1}} + t_{i-1}^2)$
$= \frac{1}{2}(t_i^2 - t_{i-1}^2)$

So $U(f,P) = \sum t_i(t_i - t_{i-1})$

$$\geq \sum A_i$$

$$= \sum \frac{1}{2}(t_i^2 - t_{i-1}^2)$$

$$= \frac{1}{2}(t_n^2 - t_0^2) \quad \text{(telescoping sum)}$$

$$= \frac{1}{2}(1^2 - 0^2)$$

$$= \frac{1}{2}$$

So $U(f,P) \geq \frac{1}{2}$, and so $\inf\{U(f,P)\} \geq \frac{1}{2}$.

Hence $\inf = \frac{1}{2}$ as desired.

A2 Q5 Suppose $b>0$ and $f(x)=x$ for all $x\in\mathbb{R}$. Show that f is integrable ~~and~~ on $[0,b]$ and $\int_0^b f=\frac{b^2}{2}$.

Sol. Posted solutions use ε-P criterion; I'll use sup-inf so you can see both.

Let's try to find partitions whose upper/lower sums are very close to $\frac{b^2}{2}$. We want a fine partition.

For each n, let $P_n=\{t_0,\dots,t_n\}$ be the partition of $[0,b]$ with n equal length subintervals (i.e. $t_i=0+\frac{ib}{n}=\frac{ib}{n}$ for each $i=0,\dots,n$.)

Then $m_i=\inf\{f(x): x\in[t_i-t_{i-1}]\}=f(t_{i-1})=t_{i-1}$
$M_i=\sup$ " " $=f(t_i)=t_i$

since f is increasing.

$$\begin{aligned}
\text{So, } L(f,P_n)&=\sum_{i=1}^{n} t_{i-1}(t_i-t_{i-1})\\
&=\sum_{i=1}^{n}\frac{(i-1)b}{n}\left(\frac{b}{n}\right)\\
&=\frac{b^2}{n^2}\sum_{i=1}^{n}(i-1)=\\
&=\frac{b^2}{n^2}\sum_{i=0}^{n-1} i=\frac{b^2}{n^2}\frac{(n-1)(n)}{2}=\frac{b^2}{2}\cdot\left(\frac{n-1}{n}\right)
\end{aligned}$$

~~Since $\frac{n-1}{n} \le 1$ for all n~~

So $\frac{b^2}{2}\cdot\left(\frac{n-1}{n}\right) \le \sup_P \{L(f,P)\}$ for ~~each~~ all n,

so $\frac{b^2}{2} \le \sup_P \{L(f,P)\}$

Similarly, one can show that

$U(f,P_n) = \frac{b^2}{2}\left(\frac{n+1}{n}\right)$

so $\cancel{\frac{b^2}{2}\left(\frac{n+1}{n}\right)}$ $\inf_P \{U(f,P)\} \le \frac{b^2}{2}\left(\frac{n+1}{n}\right)$ for all n

so $\inf_P \{U(f,P)\} \le \frac{b^2}{2}$.

always true

Thus, $\frac{b^2}{2} \le \sup\{L(f,P)\} \le \inf\{U(f,P)\} \le \frac{b^2}{2}$.

so $\sup\{L(f,P)\} = \inf\{U(f,P)\}$ and $= \frac{b^2}{2}$.

$\int_0^b f = \frac{b^2}{2}$.

~~Remark. If~~