<u>TBB #7.2.6</u> Define the <u>symmetric derivative</u> of $f$ as

$$f'_s(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}.$$ Show that

(1) $f'_s(x) = f'(x)$ when $f'(x)$ exists.

(2) $f'_s(x)$ may exist even when $f$ is not differentiable at $x$.

<u>Sol.</u> (1) Observe that $f(x+h)-f(x-h) = [f(x+h)-f(x)] + \overset{f(x)-f(x-h)}{\cancel{[f(x-h)-f(x)]}}$

So $$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{2h} - \lim_{h\to 0}\frac{f(x)-f(x-h)}{2h}$$

Let's compute the last two limits

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{2h} = \frac{1}{2}f'(x).$$

And
$$\lim_{h\to 0}\frac{f(x)-f(x-h)}{2h} = \lim_{h\to 0} f(x)-f(x+(-h)) \cancel{\frac{f(x-h)-f(x)}{2(-h)}}$$
$$= \lim_{h\to 0}\frac{f(x-h)-f(x)}{2(-h)}$$
$$= \lim_{h\to 0}\frac{f(x+(-h))-f(x)}{2(-h)}$$
$$= \lim_{h\to 0^+}\frac{f(x+(-h))-f(x)}{2(-h)}$$
$$= \lim_{h\to 0^-}\frac{f(x+h)-f(x)}{2h}$$
$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{2h}$$
$$= \frac{1}{2}f'(x)$$

So $f'_s(x) = \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x)$

<u>Note:</u> Limit manipulations assumed that the limit existed beforehand, but if we started from the bottom and worked up instead, we would know that already,

(2), Consider $f(x)=|x|$ at $x=0$.

$$f_s'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0-h)}{2h}$$
$$= \lim_{h\to 0} \frac{|h|-|-h|}{2h}$$
$$= \lim_{h\to 0} \frac{|h|-|h|}{2h}$$
$$= 0$$

However, $f(x)=|x|$ is not differentiable at $0$. ■

<u>TBB # 7.2.11</u> Give an example of a

(1) a function with an infinite derivative at some point

(2) a function $f$ with $f_+'(x_0)=\infty$ and $f_-'(x_0)=-\infty$ at some point $x_0$.

<u>Sol.</u> (1) Consider $f(x)=\sqrt{x}$ at $x=0$.

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{x}-0}{x-0} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty \quad (\mathrm{dom}(f) = [0,\infty))$$

Or: $g(x) = \begin{cases} \frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

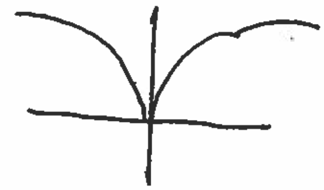
Then $\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{1}{x^2} = \infty$.

<u>Note.</u> ~~g(x) g is R→R~~ not g(x) is not continuous at $x=0$ but we calculated a derivative there. ~~Why does this not contra~~ Does this contradict differentiability $\Rightarrow$ continuity?

No, b/c differentiability means the limit is <u>finite</u>.

~~BB #7.6.4 Let f~~

(2) Let $h(x) = \sqrt{|x|} = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x} & x < 0 \end{cases}$



We have $h_+'(0) = \lim_{x\to 0^+} \cancel{f(x) - f(0)} \dfrac{\sqrt{x} - 0}{x - 0} = \lim_{x\to 0^+} \dfrac{1}{\sqrt{x}} = \infty$

$$h_-'(0) = \lim_{x\to 0^-} \frac{\sqrt{-x} - 0}{x - 0} = \lim_{x\to 0^-} -\frac{\sqrt{-x}}{-x}$$
$$= \lim_{x\to 0^-} -\frac{\sqrt{-x}}{(\sqrt{-x})^2}$$
$$= \lim_{x\to 0^-} \frac{-1}{\sqrt{-x}} = -\infty.$$ ∎

**BB #7.2.13** Let $f$ be increasing and diff. on an interval $I$.

(1) Is $f'(x) \geq 0$ on $I$?

(2) Is $f'(x) > 0$ on $I$?

Sol. Note this is asking about the converse of one of your assignment problems.

(1) We have $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ for any $a \in I$.

Observe that the expression $\dfrac{f(x) - f(a)}{x - a}$ is always $> 0$:

if ~~$x - a \to 0$, then~~ $0 < x - a$, then $a < x$ so $f(a) < f(x)$ so $f(x) - f(a) > 0$ (since $f$ increasing), so $x - a$ and $f(x) - f(a)$ have the <u>same sign</u>. Similar if $0 > x - a$.

So $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a} \geq 0$. (note $\geq$ not $>$).

So (1) is true.

(2) ~~A~~ is in fact false.

Counterexample (good to know!)

$f(x) = x^3$ is differentiable and increasing (show this).

and $f'(x) = 3x^2$, so $f'(0) = 0$. $\boxtimes$

~~TBB #7.2.15~~

Aside: Note that $f(x) = x^3$ satisfies $f'(0) = 0$, but $f$ has no local extremum at $x = 0$ so this function is also a counterexample to the converse of this statement:

$f$ diff on $I$, $x_0$ in the interior of $I$ is a local extremum $\Longrightarrow f'(x_0) = 0$

TBB #7.2.15 Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose $f$ is diff at $0$ and $f'(0)$, and that for any $x, y \in \mathbb{R}$, $f(x+y) = f(x) f(y)$. Prove that $f$ must be differentiable everywhere and $f'(x) = f(x)$.

Sol. We want to determine the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) f(h) - f(x)}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}.$$

Notice that this last limit is the difference quotient ~~lim~~ $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{h - 0}$, provided that $\boxed{f(0) = 1.}$

Can we deduce that $f(0) = 1$?

Observe that $f(0) = f(0+0) = f(0) f(0)$, so $0 = f(0)[f(0) - 1]$. So $f(0) = 0$ or $f(0) = 1$.

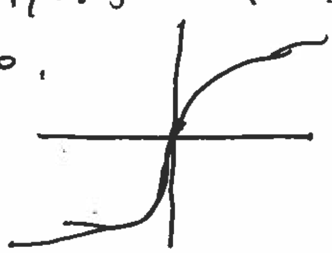
But if $f(0)=0$, then $f(x)=f(x+0)=f(x)f(0)=0$ for all $x\in\mathbb{R}$, so $f\equiv 0$ (constant function).
But then ~~f is nonconst~~ $f'(0)\neq 1$, a contradiction.
So $f(0)=1$, which we wanted to show.

Thus, $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$$
$$=f(x)f'(0)$$
$$=f(x),$$
so $f$ is differentiable everywhere with $f'(x)=f(x)$ as desired.

<u>Aside</u>: There is a <u>unique</u> function $f$ satisfying the hypothesis in this problem, namely $f(x)=e^x$. The fact that ~~f(x)~~ the function $e^x$ satisfies $f(x+y)=f(x)f(y)$ implies that $e^x$ is an isomorphism between the groups $(\mathbb{R},+)$ and $(\mathbb{R}_{>0},\times)$!

<u>BB #7.2.18</u> Suppose that $f'(x_0)=\infty$. What can you say, if anything, about continuity of $f$ at $x_0$?

<u>Sol</u>. Not much. Let $f_1(x)=x^{\frac{1}{3}}$. Then $f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$ so $f'(0)=\infty$.



. Note $f$ is continuous at $0$.

However $\cancel{f}\,f_2(x)=\begin{cases}\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ is not continuous at $0$ and $f_2'(0)=\infty$. So conclusion: $f$ may or may not be continuous at $x_0$.

**B&S 6.2.17** Let $f, g$ diff on $\mathbb{R}$ and suppose $f(0)=g(0)$ and $f'(x) \le g'(x)$ $\forall x \ge 0$. Show that $f(x) \le g(x)$ $\forall x \ge 0$.

**Sol.** Consider the function $h(x) = g(x) - f(x)$ which is diff and cont on $[0, \infty)$. We have $h(0) = 0$, and for any $x \geq 0$ $\exists\, c_x \in (0, x)$ st.

$$h'(c_x) = \frac{h(x) - h(0)}{x - 0} = \frac{g(x) - f(x)}{x}$$

But for any $x \geq 0$ $h'(x) = g'(x) - f'(x)$

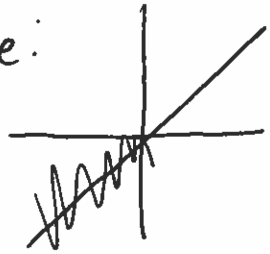
So $h'(c_x) = g'(c_x) - f'(c_x) \geq 0$.

Hence $\cancel{h(x)}$ $g(x) - f(x) = x h'(c_x) \geq 0$

So $f(x) \leq g(x)$ as desired. $\blacksquare$

TBB #7.6.4 Suppose $f'(x) \geq c > 0$ for all $x \in [0,\infty)$. Show that $\lim\limits_{x\to\infty} f(x) = \infty$.

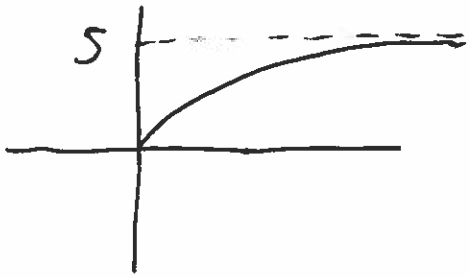
Sol. Picture:



$f(x) = x$
$f'(x) = 1 \geq 1 > 0$ for all $x \geq 0$
(Note that $f'(0) = f_+'(0) = 1$.

The condition that $\exists c$ with $f'(x) \geq c > 0$ is important. It is not enough that $f'(x) > 0$ for all $x \in [0,\infty)$.

Ex.




$f(x) = \frac{-5}{x+1} + 5$

$f'(x) = \frac{5}{(x+1)^2} > 0$ for all $x \in [0,\infty)$

But $\lim\limits_{x\to\infty} f(x) = 5$.

Let's relate $f'$ to $f$ via MVT.
For any $x \in [0,\infty)$, there exists some $a_x \in (0, x)$ such that $f'(a) = \frac{f(x) - f(0)}{x - 0}$, so $f(x) = f'(a)x + f(0)$.
But $f'(a) > c$ so $f(x) \geq cx + f(0)$
Since $\lim\limits_{x\to\infty} (cx + f(0)) = \infty$, as $c > 0$, we have $\lim\limits_{x\to\infty} f(x) = \infty$ as desired.