Crash course on logic

Dr. Darío A. Valdebenito[1]

Definition 1. A *proposition* is a statement whose truth value can be determined.

Example. The proposition "there are prime numbers" is a proposition with truth value True. "All numbers are prime" is a proposition with truth value False. "This statement is false" is not a statement.

There are statements which do not fall under this paradigm, such as the last example, which we will ignore in these notes.

Definition 2. Let p and q be propositions. We define the following operations:

(a) p AND q, denoted $p \wedge q$, which takes the value True if p and q are both True, False otherwise.

(b) p OR q, denote $p \vee q$, which takes the value True if either p or q is True, False otherwise.

(c) NOT p, denoted $\neg p$, which takes the opposite value of p.

The negation of $p \wedge q$ is given by $\neg p \vee \neg q$. The negation of $p \vee q$ is $\neg p \wedge \neg q$. You can easily verify what happens to the truth values.

Example. Let p and q be the statements $x > 1$ and $x < 2$, respectively. The proposition $p \wedge q$ is True for $x \in (1, 2)$, False otherwise. $\neg p$ and $\neg q$ are $x \leq 1$ and $x \geq 2$, respectively, and the negation of $p \wedge q$ is $\neg p \vee \neg q$, or, in plain terms, $x \leq 1$ or $x \geq 2$, which is True for any real number x such that $x \notin (1, 2)$.

Definition 3. Given p and q propositions, we define:

(a) Equivalence, denoted $p \Leftrightarrow q$, which takes the value True if p and q have the same value, False otherwise.

(b) "If p, then q," denoted $p \Rightarrow q$, which takes the value True if q is True whenever p is True. If this condition does not hold, then $p \Rightarrow q$ takes the value False.

Example. Consider the statements p and q given by "Alice is in Canada" and "Alice is in North America," respectively. Then $p \Leftrightarrow q$ is False (Alice could be in the United States, in which case p is False, but q is True), $p \Rightarrow q$ is True (if Alice is in Canada, then she *must* be in North America), and $q \Rightarrow p$ is False (Alice could be in Mexico, in which case she is in North America, but not in Canada). Note that in $p \Rightarrow q$, if p is False, then the value of q is irrelevant[2]: if Alice is not in Canada, she could be in North America or not.

In proofs, the fact that $p \Rightarrow q$ and $\neg p \vee q$ take the same truth values is useful. The negation of $p \Rightarrow q$ is thus $p \wedge \neg q$ (where we have used that $\neg\neg p = p$).

[1] Department of Mathematics and Statistics, McMaster University

[2] Mathematicians sometimes say, "false implies anything" to state this fact.

Example. Going back to our previous example, $p \Rightarrow q$ stated, "if Alice is in Canada, then she must be in North America." Here $\neg p \vee q$ reads, "Alice is not in Canada or Alice is in North America," which is True (just like the original statement). The negation of $p \Rightarrow q$ is $p \wedge \neg q$, "Alice is in Canada and is not in North America." This statement is False, which means $p \Rightarrow q$ must be True.

A deeper example is the (non-)theorem:

Theorem 4. ~~*If x_1 and x_2 are solutions of $ax = b$, then $x_1 = x_2$.*~~

The foregoing theorem is actually False, because its negation is True. Indeed, take $a = b = 0$, $x_1 = 1$, $x_2 = 2$. Then the first part of the statement is True, but the second part is False.

Another important relation for proofs is that $p \Leftrightarrow q$ is equivalent to $p \Rightarrow q$ and $q \Rightarrow p$. Equivalence is often written as "p if and only if q." This is often used in mathematical proofs as well.

Theorem 5. *Let A be a square matrix. The following statements are equivalent:*

(a) $A - \lambda I$ is singular.

(b) There is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

Proof. We first prove "if (a), then (b)." If $A - \lambda I$ is singular, then the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has infinitely many solutions. Fix any nonzero $\mathbf{v}$ solving the equation. Then

$$0 = (A - \lambda I)\mathbf{v} = A\mathbf{v}0\lambda I\mathbf{v} = A\mathbf{v} - \lambda\mathbf{v},$$

and rearranging we get $A\mathbf{v} = \lambda\mathbf{v}$.

Now we prove "if (b), then (a)." We have $A\mathbf{v} = \lambda\mathbf{v} = \lambda I\mathbf{v}$, or $A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0}$, which can be rewritten as $(A - \lambda I)\mathbf{v} = \mathbf{0}$. The last statement means that $A - \lambda I$ must be singular, since the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a nontrivial solution. □

"For all" and "exists"

A proposition p may take different truth values depending on a certain variable x, in that case we write $p(x)$. We cannot determine whether $p(x)$ is True or False without some sort of information on x. Consider, e.g., the proposition $x^2 > x$. This proposition is True if $x < 0$ or $x > 1$, False otherwise.

Definition 6. Let S be a set.

- The proposition "for all x in S, $p(x)$," denoted $(\forall x \in S)p(x)$ takes the value True if $p(x)$ is True for each and every $x \in S$, False otherwise.
- The proposition "there exists x in S, $p(x)$," denoted $(\exists x \in S)p(x)$, takes the value True if there is one $x_0 \in S$ such that $p(x_0)$ is True; if no such x_0 exists, the proposition takes the value False.

As before, we are interested in the negations of the aforementioned propositions. The negation of $(\forall x \in S)p(x)$ is $(\exists x \in S)\neg p(x)$, while the negation of $(\exists x \in S)p(x)$ is $(\forall x \in S)\neg p(x)$.

Example. Let $S = \mathbb{R}$, and the proposition $p(x)$ given by $x^2 > x$, with negation $x^2 \leq x$. Consider the proposition q given by $(\forall x \in \mathbb{R})(x^2 > x)$, and its negation is $(\exists x \in \mathbb{R})(x^2 \leq x)$. We can see that $\neg q$ is True: $\neg p(1/2)$ is the proposition $(1/2)^2 \leq 1/2$, which is True.

Remark. Notice that the proposition $(\forall x \in \mathbb{R})(x^2 > x)$ was labelled q, not $q(x)$. This is because we cannot input a value of x into the whole proposition, only into $x^2 > x$, which is why $p(x)$ is actually denoted $p(x)$.

Example. Let $S = \{x_1, \ldots, x_n\}$ be a set of n individuals, I be the set of all pairs (i, j) with $1 \leq i, j \leq n$ and $i \neq j$, and let b_j be the birthday of x_j. The statement "there are two individuals with the same birthday" can be written as

$$(\exists (i, j) \in I)(b_i = b_j).$$

Its negation is therefore

$$(\forall (i, j) \in I)(b_i \neq b_j).$$

Notice that the original statement does not exclude the possibility of three or more individuals sharing the same birthday. Note also that the condition $i \neq j$ is essential: without it, the statement is obviously true (because $b_j = b_j$).