Mathematics 3A03 — Real Analysis I

TERM TEST — 27 February 2025

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **6 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There is a blank page after questions 4, 5 and 6, and an additional blank page at the end.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) Every continuous function is differentiable.

TRUE **FALSE**

(b) For any integrable function $f : \mathbb{R} \to \mathbb{R}$, the function $F(x) = \int_0^x f$ is continuous.

TRUE FALSE

See "Integrals are continuous" on slide 53 of the integration lectures.

(c) Every differentiable function on a closed interval $[a, b]$ has a maximum and minimum value on $[a, b]$.

TRUE FALSE

(d) The instructor for this course is Taylor Swift.

TRUE **FALSE**

(e) Some integrable functions map compact sets to compact sets.

TRUE FALSE

Any continuous function does this (*cf.* Extreme Value Theorem). As an example, a constant function is integrable and maps any set to a single point. Hence, in particular, any compact set is mapped to that point, which is a compact set.

(f) If f is the second derivative of a function (*i.e.*, $f = g''$ for some function g) then f has the intermediate value property.

TRUE FALSE

[9] **QUESTION 2.** For each of the sets E in the table below, answer **YES** or **NO** in each column to indicate whether or not E is open, closed, or compact. *Do not justify your answers.*

Set E	**Open?**	**Closed?**	**Compact?**
$(0,1)\cap\mathbb{Q}$	**NO**	**NO**	**NO**
$\varnothing$	**YES**	**YES**	**YES**
$\{0\}\cup\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$	**NO**	**YES**	**YES**

[6] **QUESTION 3.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the closure ($\overline{E}$), the interior (E°), and the boundary (∂E). *Do not justify your answers.*

E	$\overline{E}$	E°	∂E
$(-\sqrt{2},\sqrt{2})$	$[-\sqrt{2},\sqrt{2}]$	E	$\{-\sqrt{2},\sqrt{2}\}$
$\left\{-\frac{1}{\sqrt{1+n^2}} : n\in\mathbb{N}\right\}$	$E\cup\{0\}$	$\varnothing$	$E\cup\{0\}$

[9] **QUESTION 4.**

[2] (a) State the formal definition of "the function f is *differentiable* at the point $c \in \mathbb{R}$".

f is defined in a neighbourhood of c and $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

[2] (b) State the *Mean Value Theorem* (MVT).

If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}\,.$$

[5] (c) Suppose $a < b$ and f is differentiable on $[a, b]$. Prove that if $f'(x) \geq M$ for all $x \in [a, b]$, then $f(b) \geq f(a) + M(b - a)$.

Proof. Since f is differentiable on $[a, b]$, it is certainly continuous on $[a, b]$ and differentiable on (a, b), so by the MVT there exists $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \geq M\,,$$

where the last inequality follows because $f'(x) \geq M$ for all $x \in [a, b]$ (hence, in particular, for $x = \xi$). Therefore, since $a < b$,

$$f(b) - f(a) \geq M(b - a)\,,$$

i.e.,

$$f(b) \geq f(a) + M(b - a)\,,$$

as required. □

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. . . Continued. . .

[10] **QUESTION 5.**

Suppose $a < c < b$ and that $f(x)$ is integrable on $[a,b]$. Prove that f is integrable on each of the two subintervals, $[a,c]$ and $[c,b]$. Show, moreover, that

$$\int_a^b f = \int_a^c f + \int_c^b f\,.$$

Proof. Since f is integrable on $[a,b]$, given any $\varepsilon > 0$ we can find a partition $P = \{t_0, \dots, t_n\}$ such that

$$U(f,P) - L(f,P) < \varepsilon\,.$$

Now let Q be the partition of $[a,b]$ that contains all the points of P and (if it is not already in P) the point c. Since $P \subseteq Q$, it follows that

$$U(f,Q) - L(f,Q) \le U(f,P) - L(f,P) < \varepsilon\,.$$

Since Q contains c, we can break it up into two parts, $Q = Q_1 \cup Q_2$, where (for some $j \in \mathbb{N}$)

$$\begin{aligned} Q_1 &= \{a, t_1, \dots, t_{j-1}, c\}\,, \\ Q_2 &= \{c, t_{j+1}, \dots, t_{n-1}, b\}\,. \end{aligned}$$

Consequently,

$$\begin{aligned} U(f,Q) &= U(f,Q_1) + U(f,Q_2)\,, \\ L(f,Q) &= L(f,Q_1) + L(f,Q_2)\,, \end{aligned}$$

and hence

$$U(f,Q) - L(f,Q) = \big[U(f,Q_1) - L(f,Q_1)\big] + \big[U(f,Q_2) - L(f,Q_2)\big].$$

But both terms in square brackets are non-negative, and hence each must be less than ε. Thus, we have found partitions (Q_1 and Q_2) of $[a,c]$ and $[c,b]$, respectively, that ensure the difference between the upper and lower sums of f for Q_i is less than ε, *i.e.*, f is, in fact, integrable on both subintervals.

Given that f is integrable on $[a,b]$, $[a,c]$ and $[c,b]$, consider any partition P of $[a,b]$ and let $Q = P \cup \{c\}$. Then Q can be subdivided into separate partitions, Q_a of $[a,c]$ and Q_b of $[c,b]$, and we have

$$\begin{aligned} L(f,Q_a) &\le \int_a^c f \le U(f,Q_a) \\ L(f,Q_b) &\le \int_c^b f \le U(f,Q_b). \end{aligned}$$

Consequently,

$$L(f,P) \le L(f,Q) \le \int_a^c f + \int_c^b f \le U(f,Q) \le U(f,P)\,.$$

This is true for *any* partition P of $[a,b]$, hence

$$\sup\big\{L(f,P) : L \text{ a partition of } [a,b]\big\} \le \int_a^c f + \int_c^b f \;\le\; \inf\big\{U(f,P) : L \text{ a partition of } [a,b]\big\}\,.$$

But since f is integrable on $[a,b]$, the sup and inf above are both equal to $\int_a^b f$, hence

$$\int_a^b f = \int_a^c f + \int_c^b f\,,$$

as required. □

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. . . *Continued*. . .

[10] **QUESTION 6.**

[2] (a) State the First Fundamental Theorem of Calculus (FFTC).

Let f be integrable on $[a, b]$, and define F on $[a, b]$ by

$$F(x) = \int_a^x f\,. \tag{1}$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c, and

$$F'(c) = f(c)\,.$$

[2] (b) State the Second Fundamental Theorem of Calculus (SFTC).

If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)\,.$$

[6] (c) Suppose f is continuous on $[a, b]$. Prove that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a)\,. \tag{$*$}$$

Proof. The trivial case ($a = b$) need not be mentioned, but it is fine to mention it: If $a = b$ then take $c = a$ and note that ($*$) says $0 = 0$. Now assume $a < b$.

Since f is continuous on $[a, b]$, it is integrable on $[a, b]$, so the FFTC implies that the function $F(x)$ in equation (1) is well-defined and differentiable on $[a, b]$, and $F'(x) = f(x)$ for all $x \in [a, b]$.

Consequently, the SFTC implies that

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{2}$$

Now, since F is differentiable on $[a, b]$ we can apply the MVT as stated in question 4(b) to conclude that there exists $c \in (a, b)$ such that

$$F(b) - F(a) = F'(c)(b-a) = f(c)(b-a) \tag{3}$$

(where $F'(c) = f(c)$ because $F'(x) = f(x)$ for all $x \in [a, b]$).

Thus, combining equations (2) and (3), we obtain equation ($*$), which is known as the *Mean Value Theorem for integrals.*

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