Mathematics 3A03 — Real Analysis I

TERM TEST — 27 February 2025

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **6 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There is a blank page after questions 4, 5 and 6, and an additional blank page at the end.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) Every continuous function is differentiable.

TRUE **FALSE**

(b) For any integrable function $f : \mathbb{R} \to \mathbb{R}$, the function $F(x) = \int_0^x f$ is continuous.

TRUE **FALSE**

(c) Every differentiable function on a closed interval $[a, b]$ has a maximum and minimum value on $[a, b]$.

TRUE **FALSE**

(d) The instructor for this course is Taylor Swift.

TRUE **FALSE**

(e) Some integrable functions map compact sets to compact sets.

TRUE **FALSE**

(f) If f is the second derivative of a function (*i.e.*, $f = g''$ for some function g) then f has the intermediate value property.

TRUE **FALSE**

[9] **QUESTION 2.** For each of the sets E in the table below, answer **YES** or **NO** in each column to indicate whether or not E is open, closed, or compact. *Do not justify your answers.*

Set E	**Open?**	**Closed?**	**Compact?**
$(0,1)\cap\mathbb{Q}$			
$\varnothing$			
$\{0\}\cup\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$			

[6] **QUESTION 3.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the closure ($\overline{E}$), the interior (E°), and the boundary (∂E). *Do not justify your answers.*

E	$\overline{E}$	E°	∂E
$(-\sqrt{2},\sqrt{2})$			
$\left\{-\frac{1}{\sqrt{1+n^2}} : n\in\mathbb{N}\right\}$			

[9] **QUESTION 4.**

[2] (a) State the formal definition of “the function f is *differentiable* at the point $c \in \mathbb{R}$”.

[2] (b) State the *Mean Value Theorem* (MVT).

[5] (c) Suppose $a < b$ and f is differentiable on $[a, b]$. Prove that if $f'(x) \geq M$ for all $x \in [a, b]$, then $f(b) \geq f(a) + M(b - a)$.

This page has been left blank to provide additional space if needed for your solution of question 4.

[10] **QUESTION 5.**

Suppose $a < c < b$ and that $f(x)$ is integrable on $[a, b]$. Prove that f is integrable on each of the two subintervals, $[a, c]$ and $[c, b]$. Show, moreover, that

$$\int_a^b f = \int_a^c f + \int_c^b f\,.$$

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. . . Continued. . .

[10] **QUESTION 6.**

[2] (a) State the First Fundamental Theorem of Calculus (FFTC).

[2] (b) State the Second Fundamental Theorem of Calculus (SFTC).

[6] (c) Suppose f is continuous on $[a, b]$. Prove that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\, dx \;=\; f(c)\,(b-a)\,. \qquad (*)$$

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