Mathematics 3A03 — Real Analysis I

TERM TEST 2 — 26 March 2026

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

INSTRUCTOR'S SOLUTIONS

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of **15 pages** (*i.e.,* **7 double-sided pages**). There are **5 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There are several blank pages at the end; if you use those pages, state which question you are answering.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) In a metric space, the union of any collection of closed sets is closed.

TRUE **[FALSE]**

(b) If $\mathcal{M}$ is a non-empty set and d is the discrete metric on $\mathcal{M}$ then every set in the metric space $(\mathcal{M}, d)$ is open.

[TRUE] **FALSE**

(c) The instructor for this course is Mark Carney.

TRUE **[FALSE]**

(d) In any metric space $(\mathcal{M}, d)$, if $x \in \mathcal{M}$ and $0 < r_1 < r_2$ then the set difference $B_{r_2}(x) \setminus B_{r_1}(x)$ is neither open nor closed.

TRUE **[FALSE]**

(e) The series $\displaystyle\sum_{k=1}^{\infty} \frac{x}{k^2}$ converges uniformly on the interval $(-\pi, \pi)$.

[TRUE] **FALSE**

(f) If $f_n : [a, b] \to \mathbb{R}$ is integrable for each n, and $f_n \to f$ uniformly, then f is integrable and $\int_a^b f_n \to \int_a^b f$.

[TRUE] **FALSE**

[8] **QUESTION 2.** Let ℓ^∞ denote the space of all bounded sequences of real numbers, with the sup norm. Thus, if $x = (x_n) \in \ell^\infty$ then

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| .$$

The subset E is defined by

$$E = (0,1)^\infty = \Big\{(x_n) \in \ell^\infty : 0 < x_n < 1 \text{ for all } n \in \mathbb{N}\Big\}.$$

[4] (a) Give an example of a point x in E°, the interior of E, and prove that $x \in E^\circ$.

Let $x = (\frac{1}{2}, \frac{1}{2}, \dots)$, *i.e.*, the constant sequence with $x_n = \frac{1}{2}$ for all n. Then $0 < x_n < 1$ for all $n \in \mathbb{N}$, so $x \in E$. We will show that $B_{1/4}(x) \subset E$.

Let $z = (z_n) \in B_{1/4}(x)$. Then $\|z - x\|_\infty < \frac{1}{4}$, so

$$|z_n - \tfrac{1}{2}| < \frac{1}{4} \qquad \forall n \in \mathbb{N}.$$

Hence

$$\frac{1}{4} < z_n < \frac{3}{4} \qquad \forall n \in \mathbb{N}.$$

Therefore $0 < z_n < 1$ for all $n \in \mathbb{N}$, so $z \in E$. Thus $B_{1/4}(x) \subset E$, so it follows that $x \in E^\circ$.

(The full interior of E is the set of sequences in E that are uniformly bounded away from 0 and 1.) □

[4] (b) Give an example of a point y in ∂E, the boundary of E, and prove that $y \in \partial E$.

Let $y = (0, 0, \dots) \in \ell^\infty$. Then $y \notin E$, but for any $\varepsilon > 0$, the open ball $B_\varepsilon(y)$ contains a point in E, e.g., $(\varepsilon/2, \varepsilon/2, \dots) \in E$, and also a point in E^c (namely y itself). Therefore, every neighborhood of y intersects both E and E^c, so $y \in \partial E$. (∂E is the set of sequences in $[0,1]^\infty$ that are not uniformly bounded away from 0 and 1.) □

Other points in $E^c \cap \partial E$ are possible, *e.g.*, $(0, \frac{1}{2}, \frac{1}{2}, \dots)$ and similar ideas.

One can also choose a point $y \in E \cap \partial E$, *e.g.*, $(\frac{1}{n+1})_{n\in\mathbb{N}}$.

(A longer question could have asked for two points in ∂E, one in E and one in E^c.)

. . . Continued. . .

[12] **QUESTION 3.**

[2] (a) State the definitions of ***pointwise convergence*** and ***uniform convergence*** for a sequence of functions $f_k : D \to \mathbb{R}$ converging to a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$.

We say that $f_k \to f$ ***pointwise*** on D if

$$\boldsymbol{\forall x \in D},\ \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \ni \forall k \geq N,\ |f_k(x) - f(x)| < \varepsilon.$$

We say that $f_k \to f$ ***uniformly*** on D if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \ni \forall k \geq N,\ \boldsymbol{\forall x \in D},\ |f_k(x) - f(x)| < \varepsilon.$$

[5] (b) Suppose that for each $k \in \mathbb{N}$, the function $f_k : D \to \mathbb{R}$ is unbounded on the domain $D \subseteq \mathbb{R}$. If $f_k \to f$ ***pointwise*** on D, must f be unbounded on D? Either prove that f must be unbounded or give a counterexample.

No, f can be bounded. A counterexample is obtained by taking $D = \mathbb{R}$ and defining

$$f_k(x) = \frac{x}{k} \qquad \forall x \in \mathbb{R},\ \forall k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, the function f_k is unbounded on $\mathbb{R}$, since for any $M > 0$, if $x > Mk$, then

$$f_k(x) = \frac{x}{k} > M.$$

However, for each fixed $x \in \mathbb{R}$,

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \frac{x}{k} = 0.$$

Thus $f_k \to f$ pointwise on $\mathbb{R}$ to the (bounded) zero function ($f(x) = 0\ \forall x \in \mathbb{R}$). □

<u>*Note*</u>*:* Another simple counterexample is to take $D = \mathbb{R}$ and define

$$f_k(x) = \begin{cases} x, & x \geq k, \\ 0, & x < k. \end{cases}$$

Then each f_k is unbounded on $\mathbb{R}$, but (again) $f_k(x) \to 0$ as $k \to \infty$ for every fixed $x \in \mathbb{R}$.

...Continued...

[5] (c) Again suppose that for each $k \in \mathbb{N}$, the function $f_k : D \to \mathbb{R}$ is unbounded on the domain $D \subseteq \mathbb{R}$. If $f_k \to f$ ***uniformly*** on D, must f be unbounded on D? Either prove that f must be unbounded or give a counterexample.

Yes, f must be unbounded on D.

Suppose, in order to derive a contradiction, that f is bounded on D. Then there exists a constant $M > 0$ such that

$$|f(x)| \leq M \qquad \forall x \in D.$$

Since $f_k \to f$ uniformly on D, if we choose $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that for all $k \geq N$,

$$|f_k(x) - f(x)| < 1 \qquad \forall x \in D.$$

Hence, for all $k \geq N$ and all $x \in D$,

$$|f_k(x)| \leq |f_k(x) - f(x)| + |f(x)| < 1 + M.$$

Thus, for every $k \geq N$, the function f_k is bounded on D. This contradicts the assumption that every f_k is unbounded on D.

Therefore f must be unbounded on D. □

...Continued...

[11] **QUESTION 4.**

[2] (a) State the Weierstrass M-test theorem for uniform convergence of a series of functions on a domain $D \subseteq \mathbb{R}$.

Let (f_k) be a sequence of functions on a domain $D \subseteq \mathbb{R}$. Suppose there exists a sequence of real numbers (M_k) such that

$$|f_k(x)| \leq M_k \qquad \forall x \in D, \forall k \in \mathbb{N},$$

and suppose that the series $\sum_{k=1}^{\infty} M_k$ converges. Then the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges uniformly on D.

(b) For each $k \in \mathbb{N}$, define $f_k : (0, \infty) \to \mathbb{R}$ via

$$f_k(x) \;=\; 2^k \sin\left(\frac{1}{3^k x}\right).$$

[4] Prove that the series $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a, \infty)$ for any $a > 0$.

Hint: You may use the fact that $|\sin t| \leq |t|$ for all $t \in \mathbb{R}$.

Let $a > 0$. If $x \in [a, \infty)$, then $x \geq a$, so using the hint we obtain

$$|f_k(x)| = 2^k \left|\sin\left(\frac{1}{3^k x}\right)\right| \leq 2^k \left|\frac{1}{3^k x}\right| = \frac{1}{x}\left(\frac{2}{3}\right)^k \leq \frac{1}{a}\left(\frac{2}{3}\right)^k.$$

Thus, for every $x \in [a, \infty)$,

$$|f_k(x)| \leq M_k \qquad \text{where} \qquad M_k = \frac{1}{a}\left(\frac{2}{3}\right)^k.$$

Since

$$\sum_{k=1}^{\infty} M_k = \frac{1}{a}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$$

is a convergent geometric series, the Weierstrass M-test implies that $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a, \infty)$. □

. . . Continued. . .

[5] (c) For the sequence (f_k) defined in part (b), prove that the series $\sum_{k=1}^{\infty} f_k(x)$ does <u>not</u> converge uniformly on $(0, \infty)$.

Hint: Use the facts that

(1) $\sin\left(\frac{\pi}{2}\right) = 1$;

(2) if $\sum_{k=1}^{\infty} f_k$ converges uniformly then $f_k \to 0$ uniformly as $k \to \infty$.

Let's first justify the second hint for clarity (this justification is not expected, since you were told to use the hint). Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ denote the nth partial sum. Suppose, in order to derive a contradiction, that $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $(0, \infty)$. Then $S_n \to S$ uniformly on $(0, \infty)$ for some function S, and hence

$$f_n = S_n - S_{n-1} \xrightarrow[\text{unif}]{n\to\infty} S - S = 0 \qquad \text{on } (0, \infty).$$

We will show that this is impossible. For each $k \in \mathbb{N}$, define

$$x_k = \frac{2}{\pi 3^k}.$$

Then $x_k > 0$, and

$$\frac{1}{3^k x_k} = \frac{1}{3^k \cdot \frac{2}{\pi 3^k}} = \frac{\pi}{2}.$$

Therefore

$$f_k(x_k) = 2^k \sin\left(\frac{\pi}{2}\right) = 2^k.$$

Hence

$$\sup_{x>0} |f_k(x)| \geq |f_k(x_k)| = 2^k.$$

In particular,

$$\sup_{x>0} |f_k(x)| \not\to 0 \qquad \text{as } k \to \infty.$$

So f_k does not converge uniformly to 0 on $(0, \infty)$, which is a contradiction.

Therefore the series $\sum_{k=1}^{\infty} f_k(x)$ does not converge uniformly on $(0, \infty)$. □

...Continued...

[13] **QUESTION 5.**

[2] (a) State the formal definition of "a metric space $(\mathcal{M}, d)$", where $\mathcal{M}$ is a non-empty set and $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$.

A ***metric space*** $(\mathcal{M}, d)$ is a non-empty set $\mathcal{M}$ together with a function $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ such that, for all $x, y, z \in \mathcal{M}$,

(a) $d(x, y) \geq 0$,

(b) $d(x, y) = 0 \iff x = y$,

(c) $d(x, y) = d(y, x)$,

(d) $d(x, y) \leq d(x, z) + d(z, y)$.

(b) Let $\widehat{\mathbb{R}}$ denote the set of real numbers $\mathbb{R}$ together with two extra "points at infinity". Thus, this set of ***extended real numbers*** is

$$\widehat{\mathbb{R}} \;=\; \mathbb{R} \cup \{-\infty, +\infty\}\,.$$

Now define a function $h : \widehat{\mathbb{R}} \to [-1, 1]$ by

$$h(x) = \frac{x}{1 + |x|} \quad \text{for } x \in \mathbb{R}, \qquad h(-\infty) = -1, \qquad h(+\infty) = 1.$$

[4] Prove that h is one-to-one on $\widehat{\mathbb{R}}$.

Proof. We will first show that h is strictly increasing (and hence one-to-one) on $\mathbb{R}$.

If $x \in \mathbb{R}$ and $x \leq 0$, then

$$h(x) = \frac{x}{1 - x} = -1 + \frac{1}{1 - x},$$

which is non-positive and strictly increasing in x $(h'(x) = 1/(1 - x)^2 > 0 \;\forall x \leq 0)$. Similarly, if $y \in \mathbb{R}$ and $y \geq 0$, then

$$h(y) = \frac{y}{1 + y} = 1 - \frac{1}{1 + y},$$

which is non-negative and strictly increasing in y. Consequently, if $x, y \in \mathbb{R}$ and $x < 0 < y$, then $h(x) < 0 < h(y)$, so again $h(x) < h(y)$. Thus, h is strictly increasing (and hence one-to-one) on $\mathbb{R}$.

Finally, for every real number x,

$$-1 < h(x) < 1,$$

whereas

$$h(-\infty) = -1 \qquad \text{and} \qquad h(+\infty) = 1.$$

Thus, no real number maps to ± 1 and $h(-\infty) \neq h(+\infty)$. Consequently, for any $x, y \in \widehat{\mathbb{R}}$, $x \neq y \implies h(x) \neq h(y)$, *i.e.*, h is one-to-one on $\widehat{\mathbb{R}}$.

Note: It wasn't stated explicitly, but if you assumed the natural ordering of $\widehat{\mathbb{R}}$, *i.e.*, $-\infty < x < +\infty$ for all $x \in \mathbb{R}$, then the above argument proves that h is strictly increasing on $\widehat{\mathbb{R}}$. $\square$

(c) Given the function h defined in part (b), define a function $d : \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} \to \mathbb{R}$ by

$$d(x, y) \quad = \quad |h(x) - h(y)| \,.$$

[2] Compute $d(-1, 1)$ and $d(-\infty, +\infty)$.

First,

$$h(-1) = \frac{-1}{1 + |-1|} = -\frac{1}{2} \qquad \text{and} \qquad h(1) = \frac{1}{1 + |1|} = \frac{1}{2}.$$

Therefore,

$$d(-1, 1) = |h(-1) - h(1)| = \left| -\frac{1}{2} - \frac{1}{2} \right| = 1.$$

Also,

$$d(-\infty, +\infty) = |h(-\infty) - h(+\infty)| = |-1 - 1| = 2.$$

[5] (d) Prove that $(\widehat{\mathbb{R}}, d)$ is a metric space.

Proof. $\widehat{\mathbb{R}}$ is non-empty by definition (it is a union of non-empty sets), and for every $x, y \in \widehat{\mathbb{R}}$, we have $h(x), h(y) \in [-1, 1]$, so $d(x, y) = |h(x) - h(y)| \in \mathbb{R}$. We now verify the metric axioms stated in part (a), which follow from the standard properties of the absolute value function on $\mathbb{R}$.

(a) For all $x, y \in \widehat{\mathbb{R}}$, $d(x, y) = |h(x) - h(y)| \geq 0$, since the absolute value of any real number is non-negative.

(b) If $d(x, y) = 0$, then $|h(x) - h(y)| = 0$, so $h(x) = h(y)$. By part (b), h is one-to-one, and therefore $x = y$. Conversely, if $x = y$, then $d(x, y) = |h(x) - h(x)| = 0$. Thus $d(x, y) = 0 \iff x = y$.

(c) For all $x, y \in \widehat{\mathbb{R}}$, $d(x, y) = |h(x) - h(y)| = |h(y) - h(x)| = d(y, x)$.

(d) Let $x, y, z \in \widehat{\mathbb{R}}$. Then

$$d(x, y) = |h(x) - h(y)| = |(h(x) - h(z)) + (h(z) - h(y))| \,.$$

Hence, by the triangle inequality for absolute values in $\mathbb{R}$,

$$d(x, y) \leq |h(x) - h(z)| + |h(z) - h(y)| = d(x, z) + d(z, y).$$

Therefore $(\widehat{\mathbb{R}}, d)$ is a metric space. □

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