Mathematics 3A03 — Real Analysis I

TERM TEST 2 — 26 March 2026

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 14 **pages** (*i.e.,* **7 double-sided pages**). There are **5 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There are several blank pages at the end; if you use those pages, state which question you are answering.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) In a metric space, the union of any collection of closed sets is closed.

TRUE **FALSE**

(b) If $\mathcal{M}$ is a non-empty set and d is the discrete metric on $\mathcal{M}$ then every set in the metric space $(\mathcal{M}, d)$ is open.

TRUE **FALSE**

(c) The instructor for this course is Mark Carney.

TRUE **FALSE**

(d) In any metric space $(\mathcal{M}, d)$, if $x \in \mathcal{M}$ and $0 < r_1 < r_2$ then the set difference $B_{r_2}(x) \setminus B_{r_1}(x)$ is neither open nor closed.

TRUE **FALSE**

(e) The series $\displaystyle\sum_{k=1}^{\infty} \frac{x}{k^2}$ converges uniformly on the interval $(-\pi, \pi)$.

TRUE **FALSE**

(f) If $f_n : [a, b] \to \mathbb{R}$ is integrable for each n, and $f_n \to f$ uniformly, then f is integrable and $\int_a^b f_n \to \int_a^b f$.

TRUE **FALSE**

[8] **QUESTION 2.** Let ℓ^∞ denote the space of all bounded sequences of real numbers, with the sup norm. Thus, if $x = (x_n) \in \ell^\infty$ then

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

The subset E is defined by

$$E = (0,1)^\infty = \Big\{(x_n) \in \ell^\infty : 0 < x_n < 1 \text{ for all } n \in \mathbb{N}\Big\}.$$

[4] (a) Give an example of a point x in E°, the interior of E, and prove that $x \in E^\circ$.

[4] (b) Give an example of a point y in ∂E, the boundary of E, and prove that $y \in \partial E$.

...Continued...

[12] **QUESTION 3.**

[2] (a) State the definitions of ***pointwise convergence*** and ***uniform convergence*** for a sequence of functions $f_k : D \to \mathbb{R}$ converging to a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$.

[5] (b) Suppose that for each $k \in \mathbb{N}$, the function $f_k : D \to \mathbb{R}$ is unbounded on the domain $D \subseteq \mathbb{R}$. If $f_k \to f$ ***pointwise*** on D, must f be unbounded on D? Either prove that f must be unbounded or give a counterexample.

. . . Continued. . .

[5] (c) Again suppose that for each $k \in \mathbb{N}$, the function $f_k : D \to \mathbb{R}$ is unbounded on the domain $D \subseteq \mathbb{R}$. If $f_k \to f$ ***uniformly*** on D, must f be unbounded on D? Either prove that f must be unbounded or give a counterexample.

. . . Continued. . .

[11] **QUESTION 4.**

[2] (a) State the Weierstrass M-test theorem for uniform convergence of a series of functions on a domain $D \subseteq \mathbb{R}$.

(b) For each $k \in \mathbb{N}$, define $f_k : (0, \infty) \to \mathbb{R}$ via

$$f_k(x) \; = \; 2^k \sin\left(\frac{1}{3^k x}\right) .$$

[4] Prove that the series $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a, \infty)$ for any $a > 0$.

Hint: You may use the fact that $|\sin t| \leq |t|$ for all $t \in \mathbb{R}$.

. . . Continued. . .

[5] (c) For the sequence (f_k) defined in part (b), prove that the series $\sum_{k=1}^{\infty} f_k(x)$ does not converge uniformly on $(0, \infty)$.

Hint: Use the facts that

(1) $\sin\left(\frac{\pi}{2}\right) = 1$;

(2) if $\sum_{k=1}^{\infty} f_k$ converges uniformly then $f_k \to 0$ uniformly as $k \to \infty$.

...Continued...

[13] **QUESTION 5.**

[2] (a) State the formal definition of "a metric space $(\mathcal{M}, d)$", where $\mathcal{M}$ is a non-empty set and $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$.

(b) Let $\widehat{\mathbb{R}}$ denote the set of real numbers $\mathbb{R}$ together with two extra "points at infinity". Thus, this set of ***extended real numbers*** is

$$\widehat{\mathbb{R}} \;=\; \mathbb{R} \cup \{-\infty, +\infty\}\,.$$

Now define a function $h : \widehat{\mathbb{R}} \to [-1, 1]$ by

$$h(x) = \frac{x}{1+|x|} \quad \text{for } x \in \mathbb{R}, \qquad h(-\infty) = -1, \qquad h(+\infty) = 1.$$

[4] Prove that h is one-to-one on $\widehat{\mathbb{R}}$.

. . . Continued. . .

(c) Given the function h defined in part (b), define a function $d : \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} \to \mathbb{R}$ by

$$d(x, y) \quad = \quad |h(x) - h(y)| \,.$$

[2] Compute $d(-1, 1)$ and $d(-\infty, +\infty)$.

[5] (d) Prove that $(\widehat{\mathbb{R}}, d)$ is a metric space.

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