Mathematics 3A03 — Real Analysis I

TERM TEST 1 — 26 February 2026

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

INSTRUCTOR'S SOLUTIONS

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 14 **pages** (*i.e.,* **7 double-sided pages**). There are **6 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There are blank pages after questions 4, 5 and 6, and additional blank pages at the end.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do* <u>*not*</u> *justify your answers.*

(a) Every integrable function is continuous.

TRUE **FALSE**

(b) If f is integrable on $[a,b]$ and $F(x) = \int_a^x f$ then F has a maximum and minimum value on $[a,b]$.

TRUE FALSE

See "Integrals are continuous" on slide 52 of the integration lectures.

(c) The instructor for this course is Bad Bunny.

TRUE **FALSE**

(d) If E has no accumulation points, then E is not closed.

TRUE **FALSE**

(e) If E is open, then $\partial E \cap E = \varnothing$.

TRUE FALSE

(f) If $E \subset \mathbb{R}$ and there is a function $f : E \to \mathbb{R}$ that is locally bounded on E then E is compact.

TRUE **FALSE**

[9] **QUESTION 2.** For each of the sets E in the table below, answer **YES** or **NO** in each column to indicate whether or not E is open, closed, or compact. *Do not justify your answers.*

Set E	**Open?**	**Closed?**	**Compact?**
$[0, \infty)$	**NO**	**YES**	**NO**
$\bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right]$	**NO**	**NO**	**NO**
$\bigcap_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right]$	**YES**	**YES**	**YES**

[6] **QUESTION 3.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the closure ($\overline{E}$), the interior (E°), and the boundary (∂E). *Do not justify your answers.*

E	$\overline{E}$	E°	∂E
$[0,1] \cap \mathbb{Q}^c$	$[0,1]$	$\varnothing$	$[0,1]$
$\bigcup_{n=1}^{\infty} \left(n, n+\frac{1}{n}\right)$	$\bigcup_{n=1}^{\infty} \left[n, n+\frac{1}{n}\right]$	E	$\bigcup_{n=1}^{\infty} \left\{n, n+\frac{1}{n}\right\}$

[9] **QUESTION 4.**

[2] (a) State the formal definition of "the function f is *differentiable* at the point $c \in \mathbb{R}$".

f is defined in a neighbourhood of c and $\displaystyle\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists.

[2] (b) State the *Mean Value Theorem* (MVT).

If f is continuous on $[a,b]$ and differentiable on (a,b) then there exists $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}\,.$$

[5] (c) Prove that $e^x \geq x+1$ for all $x \in \mathbb{R}$ by applying the Mean Value Theorem to $f(x) = e^x$.
Hint: Consider separately the cases $x = 0$, $x > 0$ and $x < 0$.

Proof. If $x = 0$ then $e^x = 1 = x+1$, so the claim is true.
For $x \neq 0$, consider the function $f(x) = e^x$ and note that $f'(x) = f(x) = e^x$ for all x.
For $x > 0$, consider the interval $[0,x]$. MVT implies that there exists $\xi \in (0,x)$ such that

$$e^\xi = \frac{e^x - e^0}{x-0} = \frac{e^x-1}{x}\,.$$

But $e^x > 1$ for all $x > 0$, so $e^\xi > 1$, and hence

$$\frac{e^x-1}{x} > 1 \quad\Longrightarrow\quad e^x - 1 > x \quad\Longrightarrow\quad e^x > x+1\,.$$

For $x < 0$, apply MVT to the interval $[x,0]$ to infer the existence of $\xi \in (x,0)$ such that

$$e^\xi = \frac{e^0 - e^x}{0-x} = \frac{1-e^x}{-x} = \frac{1-e^x}{-x}\,.$$

But $e^\xi < 1$ because $\xi < 0$, so (since $-x > 0$) we have

$$\frac{1-e^x}{-x} < 1 \quad\Longrightarrow\quad 1-e^x < -x \quad\Longrightarrow\quad e^x > x+1\,.$$

□

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. . . Continued. . .

[10] **QUESTION 5.** Let $a < b$ and suppose f is a strictly increasing function defined on $[a, b]$.

[2] (a) Prove that f is bounded on $[a, b]$.

Proof. Since f is nondecreasing on $[a, b]$, we have $f(x) \geq f(a)$ $\forall x \in [a, b]$ and $f(x) \leq f(b)$ $\forall x \in [a, b]$. Thus

$$f(a) \leq f(x) \leq f(b), \qquad \text{for all } x \in [a, b],$$

i.e., $f(a)$ is a lower bound for f on $[a, b]$ and $f(b)$ is an upper bound for f on $[a, b]$. Hence f is bounded on $[a, b]$. □

[2] (b) Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Give the definition of the upper sum $U(f, P)$ and of the lower sum $L(f, P)$.

Let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$
$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The ***lower sum*** of f for P, denoted by $L(f, P)$, is

$$L(f, P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}).$$

The ***upper sum*** of f for P, denoted by $U(f, P)$, is

$$U(f, P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}).$$

. . . *Part* (c) *on next page*. . .

[6] (c) Prove that f is integrable on $[a,b]$.

Hint: Consider an evenly spaced partition P, so $t_i - t_{i-1} = \delta$ for each i. Prove that

$$U(f,P) - L(f,P) \;\le\; \delta[f(b) - f(a)]$$

and use this to show that f is integrable.

Proof. We know from part (a) that f is bounded on $[a,b]$. It remains to show that given any $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Consider an evenly spaced partition P with $t_i - t_{i-1} = \delta$ for $i = 1,\ldots,n$. We then have

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} m_i\delta = \delta\sum_{i=1}^{n} m_i\,,$$
$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = \sum_{i=1}^{n} M_i\delta = \delta\sum_{i=1}^{n} M_i\,,$$

and hence

$$U(f,P) - L(f,P) = \delta\sum_{i=1}^{n}(M_i - m_i)\,.$$

Since f is nondecreasing on $[a,b]$, on each subinterval $[t_{i-1},t_i]$ we have $m_i = f(t_{i-1})$ and $M_i = f(t_i)$. Therefore

$$\begin{aligned} U(f,P) - L(f,P) &= \delta\sum_{i=1}^{n}\big[f(t_i) - f(t_{i-1})\big] \\ &= \delta\Big[f(t_1) - f(t_0) + f(t_2) - f(t_1) + \cdots + f(t_n) - f(t_{n-1})\Big] \\ &= \delta\big[f(t_n) - f(t_0)\big] \\ &= \delta\big[f(b) - f(a)\big]\,. \end{aligned}$$

For the general nondecreasing situation, there are two cases to consider, as follows. Since the question stated "strictly increasing", only the second case is relevant.

(i) $f(b) = f(a)$. This occurs if f is, in fact, a constant function. In this case, given $\varepsilon > 0$ choose any evenly spaced partition (*e.g.*, $P = \{a,b\}$ will do) and note from above that $U(f,P) - L(f,P) = 0 < \varepsilon$.

(ii) $f(b) > f(a)$. Given $\varepsilon > 0$, choose an evenly spaced partition P as above, but more specifically with

$$\delta = \frac{\varepsilon}{2[f(b) - f(a)]}\,.$$

Then

$$\begin{aligned} U(f,P) - L(f,P) &= \delta\big[f(b) - f(a)\big] \\ &= \frac{\varepsilon}{2[f(b) - f(a)]}\big[f(b) - f(a)\big] \\ &= \frac{\varepsilon}{2} \;<\; \varepsilon\,, \end{aligned}$$

as required. □

. . . *Continued*. . .

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[10] **QUESTION 6.** Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is integrable.

[5] (a) Prove that there exists $c \in [a,b]$ such that

$$\int_a^c f \;=\; \int_c^b f\,. \qquad (\heartsuit)$$

Hint: Define $F(x) = \int_a^x f$ and first show that $(\heartsuit)$ can be expressed in term of F.

Note: When using theorems proved in class, state them clearly.

Proof. Define F on $[a,b]$ by

$$F(x) \;=\; \int_a^x f\,. \qquad (1)$$

Then the LHS of $(\heartsuit)$ is $F(c)$. In addition, for any $c \in [a,b]$, the integral segmentation theorem states that f is integrable on $[a,c]$ and $[c,b]$ and

$$\int_a^b f \;=\; \int_a^c f + \int_c^b f \qquad \Longleftrightarrow \qquad \int_c^b f \;=\; \int_a^b f - \int_a^c f$$

so the RHS of $(\heartsuit)$ is $F(b) - F(c)$. Thus, the claim to be proved is that there is some $c \in [a,b]$ such that

$$F(c) \;=\; F(b) - F(c)$$

i.e., there exists $c \in [a,b]$ such that

$$F(c) \;=\; \frac{F(b)}{2}. \qquad (2)$$

Now note that $F(a) = 0$ by definition, so $\frac{F(b)}{2}$ is between $F(a)$ and $F(b)$ (or potentially equal to $F(a)$ or $F(b)$). The “integrals are continuous” theorem implies that F is continuous on $[a,b]$, so the Intermediate Value Theorem (IVT) implies that there exists $c \in [a,b]$ such that $F(c) = \frac{F(b)}{2}$, as required. $\square$

. . . Part (b) *on next page. . .*

[5] (b) Show, by constructing an example, that it is possible that c in part (a) might necessarily be an endpoint of the interval $[a, b]$. Specifically, take $a = 0$ and $b = 1$ and construct an integrable function f on $[0, 1]$ such that there is no $c \in (0, 1)$ for which ($\heartsuit$) holds but ($\heartsuit$) does hold for $c = 0$ and $c = 1$.

Proof. First note that if $c = a$, then $F(c) = 0$ since $F(a) = 0$ from Equation (1).

Next note that if $c = b$, then $F(c) = F(b)$, but from Equation (2) $F(c) = F(b)/2$. Thus, $F(b) = F(b)/2$, which implies $F(b) = 0$ and thus again we must have $F(c) = 0$.

So, let's construct a function f on $[a, b]$ such that its integral up to $x \in (a, b)$ is positive but vanishes at b, *i.e.*, $F(x) > 0$ for all $x \in (a, b)$ but $F(b) = 0$.

Let $a = 0$, $b = 1$ and $f(x) = 1 - 2x$. Then f is continuous and hence integrable, and

$$F(x) = \int_0^x (1 - 2t)\, dt = x - x^2\,,$$

so $F(0) = F(1) = 0$. Therefore, the condition $F(c) = F(b)/2 = F(1)/2$ implies $F(c) = 0$. But $F(x) = x(1 - x) > 0$ for all $x \in (0, 1)$, so either $c = 0$ or $c = 1$. ☐

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