Mathematics 3A03 — Real Analysis I

TERM TEST 1 — 26 February 2026

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of **14 pages** (*i.e.,* **7 double-sided pages**). There are **6 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There are blank pages after questions 4, 5 and 6, and additional blank pages at the end.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) Every integrable function is continuous.

TRUE **FALSE**

(b) If f is integrable on $[a,b]$ and $F(x) = \int_a^x f$ then F has a maximum and minimum value on $[a,b]$.

TRUE **FALSE**

(c) The instructor for this course is Bad Bunny.

TRUE **FALSE**

(d) If E has no accumulation points, then E is not closed.

TRUE **FALSE**

(e) If E is open, then $\partial E \cap E = \varnothing$.

TRUE **FALSE**

(f) If $E \subset \mathbb{R}$ and there is a function $f : E \to \mathbb{R}$ that is locally bounded on E then E is compact.

TRUE **FALSE**

[9] **QUESTION 2.** For each of the sets E in the table below, answer **YES** or **NO** in each column to indicate whether or not E is open, closed, or compact. *Do not justify your answers.*

Set E	**Open?**	**Closed?**	**Compact?**
$[0, \infty)$			
$\bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right]$			
$\bigcap_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right]$			

[6] **QUESTION 3.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the closure ($\overline{E}$), the interior (E°), and the boundary (∂E). *Do not justify your answers.*

E	$\overline{E}$	E°	∂E
$[0, 1] \cap \mathbb{Q}^c$			
$\bigcup_{n=1}^{\infty} \left(n, n + \frac{1}{n}\right)$			

[9] **QUESTION 4.**

[2] (a) State the formal definition of "the function f is *differentiable* at the point $c \in \mathbb{R}$".

[2] (b) State the *Mean Value Theorem* (MVT).

[5] (c) Prove that $e^x \geq x + 1$ for all $x \in \mathbb{R}$ by applying the Mean Value Theorem to $f(x) = e^x$.
Hint: Consider separately the cases $x = 0$, $x > 0$ and $x < 0$.

This page has been left blank to provide additional space if needed for your solution of question 4.

. . . *Continued*. . .

[10] **QUESTION 5.** Let $a < b$ and suppose f is a strictly increasing function defined on $[a, b]$.

[2] (a) Prove that f is bounded on $[a, b]$.

[2] (b) Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Give the definition of the upper sum $U(f, P)$ and of the lower sum $L(f, P)$.

...Part (c) *on next page...*

[6] (c) Prove that f is integrable on $[a, b]$.

Hint: Consider an evenly spaced partition P, so $t_i - t_{i-1} = \delta$ for each i. Prove that

$$U(f, P) - L(f, P) \leq \delta[f(b) - f(a)]$$

and use this to show that f is integrable.

. . . *Continued. . .*

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[10] **QUESTION 6.** Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is integrable.

[5] (a) Prove that there exists $c \in [a, b]$ such that

$$\int_a^c f = \int_c^b f. \qquad (\heartsuit)$$

Hint: Define $F(x) = \int_a^x f$ and first show that ($\heartsuit$) can be expressed in term of F.

Note: When using theorems proved in class, state them clearly.

. . . *Part* (b) *on next page*. . .

[5] (b) Show, by constructing an example, that it is possible that c in part (a) might necessarily be an endpoint of the interval $[a, b]$. Specifically, take $a = 0$ and $b = 1$ and construct an integrable function f on $[0, 1]$ such that there is no $c \in (0, 1)$ for which ($\heartsuit$) holds but ($\heartsuit$) does hold for $c = 0$ and $c = 1$.

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