Mathematics 3A03 — Real Analysis I

TERM TEST #2 — 1 April 2019

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

INSTRUCTOR'S SOLUTIONS

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **5 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. The final three pages are blank to provide extra space if needed.
- The first question does not require any justification for your answers. For this question, you will be assessed on your answers only. *Do not justify your answers to question 1.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[8] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is ***True*** or ***False***. *Do not justify your answers.*

(a) Every strictly increasing function has the intermediate value property.

True **False**

(b) Every differentiable function is integrable.

True **False**

(c) Every integrable function is continuous.

True **False**

(d) Every differentiable function maps compact sets to compact sets.

True **False**

(e) For any integrable function $f : \mathbb{R} \to \mathbb{R}$, the function $F(x) = \int_0^x f$ is differentiable.

True **False**

(f) If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable everywhere then $f(x) = \int_0^x g$ for some function $g : \mathbb{R} \to \mathbb{R}$.

True **False**

(g) The instructor for this course is Justin Trudeau.

True **False**

(h) If f is any function that has an essential discontinuity then f is not the derivative of a function.

True **False**

[6] **QUESTION 2.** Suppose that $\{f_n\}$ is a sequence of functions defined on $[a,b]$, and that f is another function defined on $[a,b]$.

(a) State the formal definition of "the sequence $\{f_n\}$ ***converges pointwise*** on $[a,b]$ to f".

$$\forall x \in D, \quad \forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n \geq N, \quad |f_n(x) - f(x)| < \varepsilon\,.$$

(b) State the formal definition of "the sequence $\{f_n\}$ ***converges uniformly*** on $[a,b]$ to f".

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall x \in D, \quad \text{if } n \geq N \quad \text{then} \quad |f_n(x) - f(x)| < \varepsilon\,.$$

or

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \text{if } n \geq N \quad \text{then} \quad \sup_{x \in D} \big|f_n(x) - f(x)\big| < \varepsilon\,.$$

(c) Fill in the blanks below, using each of the words "pointwise" and "uniformly" once, in such a way that the statement is always true.

If $\{f_n\}$ converges __uniformly__ to f then $\{f_n\}$ converges __pointwise__ to f.

[16] **QUESTION 3.** Suppose D is a non-empty set of real numbers.

(a) State the formal ε-δ definition of "the function $f : D \to \mathbb{R}$ is ***continuous*** at the point $c \in D$".

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in D$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.

(b) State the formal ε-δ definition of "the function $f : D \to \mathbb{R}$ is ***uniformly continuous*** on the domain D".

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x, y \in D$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

(c) State the formal ε-δ definition of "the function $f : D \to \mathbb{R}$ is ***NOT uniformly continuous*** on the domain D".

$\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x_\delta, y_\delta \in D$ such that $|x_\delta - y_\delta| < \delta$ yet $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$.

(d) Consider the domain $D = (0, \infty)$ and let $f(x) = \dfrac{1}{x}$ for all $x \in D$.

Prove that f is uniformly continuous on $[a, \infty)$ for any $a > 0$ but f is <u>not</u> uniformly continuous on D.

(*<u>Note</u>:* The following page is blank to provide additional space for your proof.)

Proof. Given $a > 0$, consider any $x, y \in [a, \infty)$ and note that since $x > a > 0$ and $y > a > 0$ we have

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| \leq \left|\frac{y - x}{a^2}\right| .$$

Therefore, given $\varepsilon > 0$, choose $\delta = a^2 \varepsilon$ and infer that $|x - y| < \delta$ implies

$$|f(x) - f(y)| \leq \left|\frac{y - x}{a^2}\right| < \frac{a^2 \varepsilon}{a^2} = \varepsilon ,$$

i.e., f is uniformly continuous on $[a, \infty)$.

This page has been left blank to provide space for your solution of question 2(d).

For the second part, consider $\varepsilon = 1$. Given $\delta > 0$, we will find $x, y > 0$ such that $0 < y - x < \delta$ and

$$\frac{1}{x} - \frac{1}{y} > 1\,.$$

If we choose $x = \delta/2$ and $y = \delta$ then $y - x = \delta/2 < \delta$ yet

$$\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} = \frac{\delta/2}{\delta^2/2} = \frac{1}{\delta}\,.$$

If $\delta \leq 1$ then we are done. For any $\delta > 1$, we just choose $x = 1/2$ and $y = 1$ instead. $\square$

[10] **QUESTION 4.**

(a) State the formal definition of "the function f is ***differentiable*** at the point $c \in \mathbb{R}$".

f is defined in a neighbourhood of c and $\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ exists.

(b) State the ***Mean Value Theorem*** (MVT).

If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}\,.$$

(c) Prove that $e^x \geq x+1$ for all $x \in \mathbb{R}$ by applying the Mean Value Theorem to $f(x) = e^x$.

Hint: Consider separately the cases $x = 0$, $x > 0$ and $x < 0$.

If $x = 0$ then $e^x = 1 = x + 1$, so the claim is true.

For $x \neq 0$, consider the function $f(x) = e^x$ and note that $f'(x) = f(x) = e^x$ for all x.

For $x > 0$, consider the interval $[0, x]$. MVT implies that there exists $\xi \in (0, x)$ such that

$$e^\xi = \frac{e^x - e^0}{x-0} = \frac{e^x - 1}{x}\,.$$

But $e^x > 1$ for all $x > 0$, so $e^\xi > 1$, and hence

$$\frac{e^x-1}{x} > 1 \quad\Longrightarrow\quad e^x - 1 > x \quad\Longrightarrow\quad e^x > x+1\,.$$

For $x < 0$, apply MVT to the interval $[x, 0]$ to infer the existence of $\xi \in (x, 0)$ such that

$$e^\xi = \frac{e^0 - e^x}{0-x} = \frac{1-e^x}{-x} = \frac{1-e^x}{-x}\,.$$

But $e^\xi < 1$ because $\xi < 0$, so (since $-x > 0$) we have

$$\frac{1-e^x}{-x} < 1 \quad\Longrightarrow\quad 1-e^x < -x \quad\Longrightarrow\quad e^x > x+1\,.$$

$\square$

[10] **QUESTION 5.**

Suppose $a < c_1 < \cdots < c_n < b$ and $f(x) = 0$ on $[a, b]$ *except* for $x \in \{c_1, \ldots, c_n\}$. Prove that f is integrable on $[a, b]$ and find $\int_a^b f$.

Hint: First consider the case $n = 1$.

Proof. Suppose $n = 1$ so $f(x) = 0$ on $[a, c_1) \cup (c_1, b]$ and $f(c_1) \neq 0$. Suppose more specifically that $f(c_1) > 0$ (the argument is similar if $f(c_1) < 0$).

If $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$ then any subinterval $[t_{k-1}, t_k]$ contains a point other than c_1 so $L(f, P) = 0$. Therefore, if f is integrable, it must be that $\int_a^b f = 0$. We will be done if, given $\varepsilon > 0$, we can construct a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) < \varepsilon$.

Consider a partition $Q_\delta = \{a,\, c_1 - \delta,\, c_1 + \delta,\, b\}$, where δ is smaller than the distance from c_1 to either a or b. Thus $\delta < \min\{c_1 - a,\, b - c_1\}$. Since $f(x) = 0$ except at c_1, and $f(c_1) > 0$, it follows that

$$U(f, Q_\delta) = f(c_1)\big((c_1 + \delta) - (c_1 - \delta)\big) = 2 \cdot \delta \cdot f(c_1)\,.$$

Therefore, given $\varepsilon > 0$, let

$$\delta(\varepsilon) = \frac{1}{2} \min\left\{c_1 - a,\, b - c_1,\, \frac{\varepsilon}{3f(c_1)}\right\},$$

and define $P_\varepsilon = Q_{\delta(\varepsilon)}$. Then

$$U(f, P_\varepsilon) \;\leq\; 2 \cdot \delta(\varepsilon) \cdot f(c_1) \;\leq\; 2 \cdot \frac{\varepsilon}{3f(c_1)} \cdot f(c_1) \;=\; \frac{2}{3}\varepsilon \;<\; \varepsilon\,,$$

as required.

For $n > 1$ we can proceed by induction. Given $a < c_1 < \cdots < c_n < b$, let $d = (c_1 + c_2)/2$. By the induction hypothesis (that the result is true if n is replaced by $n - 1$), f is integrable on $[a, d]$ and $[d, b]$, and $\int_a^d f = \int_d^b f = 0$. Hence, by the integral segmentation theorem, f is integrable on $[a, b]$ and $\int_a^b f = 0$. □

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