Mathematics 3A03 — Real Analysis I

TERM TEST #2 — 1 April 2019

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **5 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. The final three pages are blank to provide extra space if needed.
- The first question does not require any justification for your answers. For this question, you will be assessed on your answers only. *Do not justify your answers to question 1.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[8] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is ***True*** or ***False***. *Do not justify your answers.*

(a) Every strictly increasing function has the intermediate value property.

True **False**

(b) Every differentiable function is integrable.

True **False**

(c) Every integrable function is continuous.

True **False**

(d) Every differentiable function maps compact sets to compact sets.

True **False**

(e) For any integrable function $f : \mathbb{R} \to \mathbb{R}$, the function $F(x) = \int_0^x f$ is differentiable.

True **False**

(f) If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable everywhere then $f(x) = \int_0^x g$ for some function $g : \mathbb{R} \to \mathbb{R}$.

True **False**

(g) The instructor for this course is Justin Trudeau.

True **False**

(h) If f is any function that has an essential discontinuity then f is not the derivative of a function.

True **False**

[6] **QUESTION 2.** Suppose that $\{f_n\}$ is a sequence of functions defined on $[a, b]$, and that f is another function defined on $[a, b]$.

(a) State the formal definition of "the sequence $\{f_n\}$ ***converges pointwise*** on $[a, b]$ to f".

(b) State the formal definition of "the sequence $\{f_n\}$ ***converges uniformly*** on $[a, b]$ to f".

(c) Fill in the blanks below, using each of the words "pointwise" and "uniformly" once, in such a way that the statement is always true.

If $\{f_n\}$ converges ______________ to f then $\{f_n\}$ converges ______________ to f.

[16] **QUESTION 3.** Suppose D is a non-empty set of real numbers.

(a) State the formal ε-δ definition of "the function $f : D \to \mathbb{R}$ is ***continuous*** at the point $c \in D$".

(b) State the formal ε-δ definition of "the function $f : D \to \mathbb{R}$ is ***uniformly continuous*** on the domain D".

(c) State the formal ε-δ definition of "the function $f : D \to \mathbb{R}$ is ***NOT uniformly continuous*** on the domain D".

(d) Consider the domain $D = (0, \infty)$ and let $f(x) = \dfrac{1}{x}$ for all $x \in D$.

Prove that f is uniformly continuous on $[a, \infty)$ for any $a > 0$ but f is not uniformly continuous on D.

(*Note:* The following page is blank to provide additional space for your proof.)

This page has been left blank to provide space for your solution of question 2(d).

[10] **QUESTION 4.**

(a) State the formal definition of "the function f is ***differentiable*** at the point $c \in \mathbb{R}$".

(b) State the ***Mean Value Theorem*** (MVT).

(c) Prove that $e^x \geq x + 1$ for all $x \in \mathbb{R}$ by applying the Mean Value Theorem to $f(x) = e^x$.

Hint: Consider separately the cases $x = 0$, $x > 0$ and $x < 0$.

[10] **QUESTION 5.**

Suppose $a < c_1 < \cdots < c_n < b$ and $f(x) = 0$ on $[a, b]$ *except* for $x \in \{c_1, \ldots, c_n\}$.

Prove that f is integrable on $[a, b]$ and find $\int_a^b f$.

Hint: First consider the case $n = 1$.

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