Mathematics 3A03 — Real Analysis I

<u>TERM TEST #1 — 4 March 2019</u>

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

INSTRUCTOR'S SOLUTIONS

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **7 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. The final four pages are blank to provide extra space if needed.
- The first 4 questions do not require any justification for your answers. For these, you will be assessed on your answers only. *Do <u>not</u> justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[3] **QUESTION 1.** (*Circle the correct answer.*) For each of the following sets, determine whether it is ***Countable*** or ***Uncountable***. *Do not justify your answers.*

(a) $\mathbb{R} \setminus \mathbb{Q}$

Countable **Uncountable**

(b) $\mathbb{Z} \times \mathbb{Q}$

Countable **Uncountable**

(c) $\mathbb{N} \cup \mathbb{Q}$

Countable **Uncountable**

[5] **QUESTION 2.** (*Circle the correct answer.*) Determine whether each the following statements is ***True*** or ***False***. *Do not justify your answers.*

(a) Every non-empty subset of $\mathbb{Q}$ has a least element.

True **False**

(b) Every bounded sequence of real numbers converges.

True **False**

(c) Every convergent sequence of real numbers is a Cauchy sequence.

True **False**

(d) Every Cauchy sequence of real numbers is monotonic.

True **False**

(e) Every surjective function $f : \mathbb{R} \to \mathbb{R}$ is a bijection.

True **False**

[9] **QUESTION 3.** For each of the sets E in the table below, answer ***YES*** or ***NO*** in each column to indicate whether or not E is open, dense in $\mathbb{R}$, or compact. *Do not justify your answers.*

Set E	**Open?**	**Dense in** $\mathbb{R}$**?**	**Compact?**
$(0,1)\cap\mathbb{Q}$	**NO**	**NO**	**NO**
$\varnothing$	**YES**	**NO**	**YES**
$\{0\}\cup\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$	**NO**	**NO**	**YES**

[6] **QUESTION 4.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the least upper bound ($\sup(E)$), the interior (E°), and the boundary (∂E). If the requested point or set does not exist, then indicate this with the symbol $\nexists$. *Do not justify your answers.*

E	$\sup(E)$	E°	∂E
$(-\sqrt{2},\sqrt{2})$	$\sqrt{2}$	E	$\{-\sqrt{2},\sqrt{2}\}$
$\left\{-\frac{1}{\sqrt{1+n^2}} : n\in\mathbb{N}\right\}$	0	$\varnothing$	$E\cup\{0\}$

[10] **QUESTION 5.**

(a) State the formal definition of "the sequence $\{s_n\}$ converges to L as $n \to \infty$".

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|s_n - L| < \varepsilon$.

(b) Suppose $\{a_n\}$ is a sequence of real numbers that converges to a as $n \to \infty$. Use the formal definition to prove that the sequence $\left\{a_n + \dfrac{1}{n^2}\right\}$ also converges to a as $n \to \infty$.

We must show that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\left|a_n + \frac{1}{n^2} - a\right| < \varepsilon$.
Since $a_n \to a$ as $n \to \infty$, given $\varepsilon > 0$, we can find $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$,

$$|a_n - a| < \frac{\varepsilon}{2}.$$

Now note that by the triangle inequality,

$$\left|a_n + \frac{1}{n^2} - a\right| \leq |a_n - a| + \frac{1}{n^2},$$

so if we can ensure that $1/n^2 < \varepsilon/2$, we'll be done. To that end, note that

$$\frac{1}{n^2} < \frac{\varepsilon}{2} \iff n^2 > \frac{2}{\varepsilon} \iff n > \sqrt{\frac{2}{\varepsilon}}.$$

Therefore, let

$$N_2 = \left\lceil \sqrt{\frac{2}{\varepsilon}} \right\rceil + 1,$$

from which it follows that $\forall n \geq N_2$, $n > \sqrt{\frac{2}{\varepsilon}}$, and by reversing the above steps, we have $\frac{1}{n^2} < \frac{\varepsilon}{2}$ for all $n \geq N_2$.
Finally, let $N = \max\{N_1, N_2\}$. Then, $\forall n \geq N$, we have

$$\left|a_n + \frac{1}{n^2} - a\right| \leq |a_n - a| + \frac{1}{n^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as required. □

[10] **QUESTION 6.**

(a) (*Fill in the blanks.*)

The *Bolzano-Weierstrass theorem* (BWT) states that every ___`bounded`___ sequence of real numbers contains a ___`convergent`___ subsequence.

(b) *Prove or disprove:* If $\{a_n\}$ and $\{b_n\}$ are both bounded then $\{a_n b_n\}$ contains a convergent subsequence.

This is true. Since $\{a_n\}$ is bounded, $\exists M_a > 0$ such that $|a_n| \le M_a$ for all $n \in \mathbb{N}$. Similarly, since $\{b_n\}$ is bounded, $\exists M_b > 0$ such that $|b_n| \le M_b$ for all $n \in \mathbb{N}$. Consequently, for all $n \in \mathbb{N}$ we have

$$|a_n b_n| \le |a_n|\,|b_n| \le M_a M_b \,,$$

i.e., the sequence $\{a_n b_n\}$ is bounded (by $M_a M_b$).

Hence, by BWT, $\{a_n b_n\}$ contains a convergent subsequence. □

(c) *Prove or disprove:* If $\{a_n\}$ contains a divergent subsequence and $\{b_n\}$ contains a divergent subsequence then $\{a_n b_n\}$ diverges.

This is false. Let $a_n = b_n = (-1)^n$. Then both a_n and b_n have a divergent subsequence (the entire sequence is an example), but $a_n b_n = (-1)^{2n} = 1$ for all $n \in \mathbb{N}$, so $\{a_n b_n\}$ is a constant sequence and therefore converges. □

[7] **QUESTION 7.**

(a) State one of the three equivalent conditions that can be used to define compact subsets of $\mathbb{R}$, and

The three equivalent definitions of compactness discussed in class are

(1) A is compact iff A is closed and bounded;

(2) A is compact iff every sequence in A contains a subsequence that converges to a point in A (the Bolzano-Weierstrass property);

(3) A is compact iff every open cover of A contains a finite subcover (the Heini-Borel property).

(b) use the property you stated in part (a) to prove that if A and B are both non-empty, compact subsets of $\mathbb{R}$ then $A \cup B$ is also compact.

closed and bounded: Recall that E' refers to the set of accumulation points of a set E, and E is closed iff $E' \subseteq E$.

$A \cup B$ **is closed:** Let $x \in (A \cup B)'$. We must show that $x \in A \cup B$. If $x \in A'$ then $x \in A$ because A is closed; hence $x \in A \cup B$. Similarly, if $x \in B'$ then $x \in B$ because B is closed; hence $x \in A \cup B$. If $x \notin A'$ and $x \notin B'$ then there is a deleted neighbourhood of x that contains no points of A and no points of B, *i.e.,* no points of $A \cup B$, contradicting the fact that x is an accumulation point of $A \cup B$. Thus, either $x \in A'$ or $x \in B'$ (or both), which we have seen implies that $x \in A \cup B$.

$A \cup B$ **is bounded:** Since A is bounded, there exists $M_A > 0$ such that $\forall x \in A$, $|x| < M_A$. Similarly, $\exists M_B > 0 \ni \forall x \in B\ |x| < M_B$. Therefore, any $x \in A \cup B$ satisfies $|x| < M \equiv \max(M_A, M_B)$, *i.e.,* $A \cup B$ is bounded. □

Bolzano-Weierstrass : Let $\{x_n\}$ be a sequence in $A \cup B$. We must show that $\{x_n\}$ contains a subsequence that converges to a point in $A \cup B$. Since there are infinitely many points in $\{x_n\}$, there must be infinitely points of $\{x_n\}$ in at least one of A or B, *i.e.,* $\{x_n\}$ must contain a subsequence that is either strictly in A or strictly in B. Suppose $\{x_n\}$ has a subsequence $\{a_n\} \subseteq A$. Since A is compact, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a point $a \in A$. But $\{a_{n_k}\} \subseteq A \subseteq A \cup B$, so $\{a_{n_k}\}$ is a subsequence of $\{x_n\}$ that converges to a point $a \in A \subseteq A \cup B$. □

Heini-Borel : Let $\mathcal{U}$ be an open cover of $A \cup B$. We must show that $\mathcal{U}$ contains a finite subcover of $A \cup B$. Since $\mathcal{U}$ covers $A \cup B$, it certainly covers A, and since A is compact, $\mathcal{U}$ contains a finite subcover of A, say $\{U_1, \ldots, U_n\}$. Similarly, $\mathcal{U}$ covers B, so it contains finite subcover of B, say $\{V_1, \ldots, V_m\}$. Therefore,

$$\{U_1, \ldots, U_n\} \bigcup \{V_1, \ldots, V_m\}$$

is a finite subcollection of $\mathcal{U}$ that covers $A \cup B$. □

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