Mathematics 3A03 — Real Analysis I

TERM TEST #1 — 4 March 2019

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **7 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. The final four pages are blank to provide extra space if needed.
- The first 4 questions do not require any justification for your answers. For these, you will be assessed on your answers only. *Do not justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

GOOD LUCK and ENJOY!

MARKS

[3] **QUESTION 1.** (*Circle the correct answer.*) For each of the following sets, determine whether it is ***Countable*** or ***Uncountable***. *Do not justify your answers.*

(a) $\mathbb{R} \setminus \mathbb{Q}$

Countable **Uncountable**

(b) $\mathbb{Z} \times \mathbb{Q}$

Countable **Uncountable**

(c) $\mathbb{N} \cup \mathbb{Q}$

Countable **Uncountable**

[5] **QUESTION 2.** (*Circle the correct answer.*) Determine whether each the following statements is ***True*** or ***False***. *Do not justify your answers.*

(a) Every non-empty subset of $\mathbb{Q}$ has a least element.

True **False**

(b) Every bounded sequence of real numbers converges.

True **False**

(c) Every convergent sequence of real numbers is a Cauchy sequence.

True **False**

(d) Every Cauchy sequence of real numbers is monotonic.

True **False**

(e) Every surjective function $f : \mathbb{R} \to \mathbb{R}$ is a bijection.

True **False**

[9] **QUESTION 3.** For each of the sets E in the table below, answer ***YES*** or ***NO*** in each column to indicate whether or not E is open, dense in $\mathbb{R}$, or compact. *Do not justify your answers.*

Set E	**Open?**	**Dense in** $\mathbb{R}$**?**	**Compact?**
$(0,1)\cap\mathbb{Q}$			
$\varnothing$			
$\{0\}\cup\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$			

[6] **QUESTION 4.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the least upper bound ($\sup(E)$), the interior (E°), and the boundary (∂E). If the requested point or set does not exist, then indicate this with the symbol $\nexists$. *Do not justify your answers.*

E	$\sup(E)$	E°	∂E
$(-\sqrt{2},\sqrt{2})$			
$\left\{-\frac{1}{\sqrt{1+n^2}}:n\in\mathbb{N}\right\}$			

[10] **QUESTION 5.**

(a) State the formal definition of "the sequence $\{s_n\}$ converges to L as $n \to \infty$".

(b) Suppose $\{a_n\}$ is a sequence of real numbers that converges to a as $n \to \infty$. Use the formal definition to prove that the sequence $\left\{a_n + \frac{1}{n^2}\right\}$ also converges to a as $n \to \infty$.

[10] **QUESTION 6.**

(a) (*Fill in the blanks.*)

The *Bolzano-Weierstrass theorem* (BWT) states that every ______________ sequence of real numbers contains a ______________ subsequence.

(b) *Prove or disprove:* If $\{a_n\}$ and $\{b_n\}$ are both bounded then $\{a_n b_n\}$ contains a convergent subsequence.

(c) *Prove or disprove:* If $\{a_n\}$ contains a divergent subsequence and $\{b_n\}$ contains a divergent subsequence then $\{a_n b_n\}$ diverges.

[7] **QUESTION 7.**

(a) State one of the three equivalent conditions that can be used to define compact subsets of $\mathbb{R}$, and

(b) use the property you stated in part (a) to prove that if A and B are both non-empty, compact subsets of $\mathbb{R}$ then $A \cup B$ is also compact.

This page has been left blank intentionally to provide extra space if needed.

This page has been left blank intentionally to provide extra space if needed.

This page has been left blank intentionally to provide extra space if needed.

This page has been left blank intentionally to provide extra space if needed.

THE END