Mathematics 3A03 — Real Analysis I

TERM TEST #2 — 26 November 2019

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

INSTRUCTOR'S SOLUTIONS

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **5 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There is a blank page after questions 2, 4 and 5, and an additional blank page at the end.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.
- Please *carefully* remove the staple from your test before handing it in.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) Every continuous function is differentiable.

TRUE $\boxed{\textbf{FALSE}}$

(b) Some integrable functions map compact sets to compact sets.

$\boxed{\textbf{TRUE}}$ **FALSE**

(c) For any integrable function $f : \mathbb{R} \to \mathbb{R}$, the function $F(x) = \int_0^x f$ is continuous.

$\boxed{\textbf{TRUE}}$ **FALSE**

(d) Every differentiable function on a closed interval $[a, b]$ has a maximum and minimum value on $[a, b]$.

$\boxed{\textbf{TRUE}}$ **FALSE**

(e) The instructor for this course is Neil Armstrong.

TRUE $\boxed{\textbf{FALSE}}$

(f) If f is the second derivative of a function (*i.e.*, $f = g''$ for some function g) then f has the intermediate value property.

$\boxed{\textbf{TRUE}}$ **FALSE**

[12] **QUESTION 2.** Suppose $a < b$ and consider the interval $I = (a, b)$.

[2] (a) State the formal ε-δ definition of "the function $f : I \to \mathbb{R}$ is ***continuous*** at the point $c \in I$".

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in I$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.

[2] (b) State the formal ε-δ definition of "the function $f : I \to \mathbb{R}$ is ***uniformly continuous*** on the interval I".

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

[8] (c) Consider the interval $I = (0, 1)$, and suppose $f : I \to \mathbb{R}$ is uniformly continuous on I. In addition, define $g : I \to \mathbb{R}$ via

$$g(x) = f(x) + x \,, \qquad \text{for all } x \in I.$$

Prove directly from the formal ε-δ definition that g is uniformly continuous on I.

Given $\varepsilon > 0$, since f is uniformly continuous on I, we can choose $\delta_f > 0$ such that $\forall x, y \in I$ for which $|x - y| < \delta_f$, we will have $|f(x) - f(y)| < \frac{\varepsilon}{2}$.

Having found δ_f, let $\delta = \min\{\delta_f, \frac{\varepsilon}{2}\}$. Then if $x, y \in I$ and $|x - y| < \delta$ then $|x - y| < \frac{\varepsilon}{2}$.

Now note that if $x, y \in I$ and $|x - y| < \delta$ then

$$\begin{aligned} |g(x) - g(y)| &= |f(x) + x - f(y) - y| = |f(x) - f(y) + x - y| \\ &\leq |f(x) - f(y)| + |x - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

as required. $\square$

This page has been left blank to provide space for your solution of question 2(c) if needed.

. . . *Continued*. . .

[10] **QUESTION 3.**

[3] (a) State the formal definition of "the function f is ***differentiable*** at the point $c \in \mathbb{R}$".

f is defined in a neighbourhood of c and $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists.

[3] (b) State the ***Mean Value Theorem*** (MVT).

If f is continuous on $[a, b]$ and differentiable on (a, b) then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}\,.$$

[4] (c) Suppose $a < b$ and f is differentiable on $[a, b]$. Prove that if $f'(x) \geq M$ for all $x \in [a, b]$, then $f(b) \geq f(a) + M(b - a)$.

Proof. Since f is differentiable on $[a, b]$, it is certainly continuous on $[a, b]$ and differentiable on (a, b), so by the MVT there exists $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \geq M\,,$$

where the last inequality follows because $f'(x) \geq M$ for all $x \in [a, b]$ (hence, in particular, for $x = \xi$). Therefore, since $a < b$,

$$f(b) - f(a) \geq M(b - a)\,,$$

i.e.,

$$f(b) \geq f(a) + M(b - a)\,,$$

as required. □

[12] **QUESTION 4.**

Suppose $a < c < b$ and that $f(x)$ is integrable on $[a, b]$. Prove that f is integrable on each of the two subintervals, $[a, c]$ and $[c, b]$. Show, moreover, that

$$\int_a^b f = \int_a^c f + \int_c^b f\,.$$

Proof. Since f is integrable on $[a, b]$, given any $\varepsilon > 0$ we can find a partition $P = \{t_0, \ldots, t_n\}$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

Now let Q be the partition of $[a, b]$ that contains all the points of P and (if it is not already in P) the point c. Since $P \subseteq Q$, it follows that

$$U(f, Q) - L(f, Q) \le U(f, P) - L(f, P) < \varepsilon\,.$$

Since Q contains c, we can break it up into two parts, $Q = Q_1 \cup Q_2$, where (for some $j \in \mathbb{N}$)

$$Q_1 = \{a, t_1, \ldots, t_{j-1}, c\}\,,$$
$$Q_2 = \{c, t_{j+1}, \ldots, t_{n-1}, b\}\,.$$

Consequently,

$$U(f, Q) = U(f, Q_1) + U(f, Q_2)\,,$$
$$L(f, Q) = L(f, Q_1) + L(f, Q_2)\,,$$

and hence

$$U(f, Q) - L(f, Q) = \Big[U(f, Q_1) - L(f, Q_1)\Big] + \Big[U(f, Q_2) - L(f, Q_2)\Big].$$

But both terms in square brackets are non-negative, and hence each must be less than ε. Thus, we have found partitions (Q_1 and Q_2) of $[a, c]$ and $[c, b]$, respectively, that ensure the difference between the upper and lower sums of f for Q_i is less than ε, *i.e.*, f is, in fact, integrable on both subintervals.

Given that f is integrable on $[a, b]$, $[a, c]$ and $[c, b]$, consider any partition P of $[a, b]$ and let $Q = P \cup \{c\}$. Then Q can be subdivided into separate partitions, Q_a of $[a, c]$ and Q_b of $[c, b]$, and we have

$$L(f, Q_a) \le \int_a^c f \le U(f, Q_a)$$
$$L(f, Q_b) \le \int_c^b f \le U(f, Q_b).$$

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Consequently,

$$L(f,P) \le L(f,Q) \le \int_a^c f + \int_c^b f \le U(f,Q) \le U(f,P)\,.$$

This is true for *any* partition P of $[a,b]$, hence

$$\sup\Big\{L(f,P) : L \text{ a partition of } [a,b]\Big\} \\ \le \int_a^c f + \int_c^b f \;\le\; \inf\Big\{U(f,P) : L \text{ a partition of } [a,b]\Big\}\,,$$

i.e.,

$$\int_a^b f = \int_a^c f + \int_c^b f\,,$$

as required. $\square$

...Continued...

[10] **QUESTION 5.** Suppose that $\{f_n\}$ is a sequence of functions defined on $[a, b]$, and that f is another function defined on $[a, b]$.

[2] (a) State the formal definition of "the sequence $\{f_n\}$ ***converges pointwise*** on $[a, b]$ to f".

$$\forall x \in [a, b], \quad \forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n \geq N, \quad |f_n(x) - f(x)| < \varepsilon\,.$$

[2] (b) State the formal definition of "the sequence $\{f_n\}$ ***converges uniformly*** on $[a, b]$ to f".

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \forall x \in [a, b], \quad \text{if } n \geq N \quad \text{then} \quad |f_n(x) - f(x)| < \varepsilon\,.$$

or

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad \text{if } n \geq N \quad \text{then} \quad \sup_{x \in [a,b]} \Big| f_n(x) - f(x) \Big| < \varepsilon\,.$$

[6] (c) Consider the following proposition and circle **TRUE** or **FALSE**. Support your claim with either a proof or a counterexample (*there is space on the next page*).

If each f_n is continuous on $[a, b]$ and $\{f_n\}$ converges pointwise to f then f is continuous on $[a, b]$.

TRUE $\boxed{\textbf{FALSE}}$

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Consider the sequence of continuous functions on $[0,2]$ defined by

$$f_n(x) = \begin{cases} x^n & 0 \leq x < 1, \\ 1 & 1 \leq x \leq 2. \end{cases}$$

The pointwise limit of this sequence is the function

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$$

This function f is not continuous at $x = 1$. Hence the proposition is false. $\square$

. . . Continued. . .

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