Mathematics 3A03 — Real Analysis I

TERM TEST #2 — 26 November 2019

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ____.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **5 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There is a blank page after questions 2, 4 and 5, and an additional blank page at the end.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.
- Please *carefully* remove the staple from your test before handing it in.

GOOD LUCK and ENJOY!

MARKS

[6] **QUESTION 1.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) Every continuous function is differentiable.

TRUE **FALSE**

(b) Some integrable functions map compact sets to compact sets.

TRUE **FALSE**

(c) For any integrable function $f : \mathbb{R} \to \mathbb{R}$, the function $F(x) = \int_0^x f$ is continuous.

TRUE **FALSE**

(d) Every differentiable function on a closed interval $[a, b]$ has a maximum and minimum value on $[a, b]$.

TRUE **FALSE**

(e) The instructor for this course is Neil Armstrong.

TRUE **FALSE**

(f) If f is the second derivative of a function (*i.e.*, $f = g''$ for some function g) then f has the intermediate value property.

TRUE **FALSE**

[12] **QUESTION 2.** Suppose $a < b$ and consider the interval $I = (a, b)$.

[2] (a) State the formal ε-δ definition of "the function $f : I \to \mathbb{R}$ is ***continuous*** at the point $c \in I$".

[2] (b) State the formal ε-δ definition of "the function $f : I \to \mathbb{R}$ is ***uniformly continuous*** on the interval I".

[8] (c) Consider the interval $I = (0, 1)$, and suppose $f : I \to \mathbb{R}$ is uniformly continuous on I. In addition, define $g : I \to \mathbb{R}$ via

$$g(x) = f(x) + x\,, \qquad \text{for all } x \in I.$$

Prove directly from the formal ε-δ definition that g is uniformly continuous on I.

This page has been left blank to provide space for your solution of question 2(c) if needed.

. . . *Continued*. . .

[10] **QUESTION 3.**

[3] (a) State the formal definition of "the function f is ***differentiable*** at the point $c \in \mathbb{R}$".

[3] (b) State the ***Mean Value Theorem*** (MVT).

[4] (c) Suppose $a < b$ and f is differentiable on $[a, b]$. Prove that if $f'(x) \geq M$ for all $x \in [a, b]$, then $f(b) \geq f(a) + M(b - a)$.

[12] **QUESTION 4.**

Suppose $a < c < b$ and that $f(x)$ is integrable on $[a, b]$. Prove that f is integrable on each of the two subintervals, $[a, c]$ and $[c, b]$. Show, moreover, that

$$\int_a^b f = \int_a^c f + \int_c^b f\,.$$

This page has been left blank intentionally to provide extra space for your solution of question 4 if needed.

. . . *Continued*. . .

[10] **QUESTION 5.** Suppose that $\{f_n\}$ is a sequence of functions defined on $[a, b]$, and that f is another function defined on $[a, b]$.

[2] (a) State the formal definition of "the sequence $\{f_n\}$ ***converges pointwise*** on $[a, b]$ to f".

[2] (b) State the formal definition of "the sequence $\{f_n\}$ ***converges uniformly*** on $[a, b]$ to f".

[6] (c) Consider the following proposition and circle **TRUE** or **FALSE**. Support your claim with either a proof or a counterexample (*there is space on the next page*).

If each f_n is continuous on $[a, b]$ and $\{f_n\}$ converges pointwise to f then f is continuous on $[a, b]$.

TRUE **FALSE**

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