Mathematics 3A03 — Real Analysis I

TERM TEST #1 — 29 October 2019

Duration: 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ______________________________.

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **7 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There is one blank page after question 6 and an additional three blank pages at the end.
- The first 4 questions do not require any justification for your answers. For these, you will be assessed on your answers only. *Do not justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 60.

GOOD LUCK and ENJOY!

MARKS

[2] **QUESTION 1.** (*Circle the correct answer.*) For each of the following sets, determine whether it is ***Countable*** or ***Uncountable***. *Do not justify your answers.*

(a) $\mathbb{N} \cap \mathbb{R}$ [Countable]   **Uncountable**

(b) $\left\{2^{k/2} k^n : n \in \mathbb{N},\ k \in \mathbb{Z}\right\}$ [Countable]   **Uncountable**

[6] **QUESTION 2.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) If $A \subseteq \mathbb{Q}$ is bounded and $A \neq \varnothing$ then A has a least upper bound that is a rational number.

TRUE   [**FALSE**]

(b) Every non-empty subset of $\mathbb{N}$ is bounded below.

[**TRUE**]   **FALSE**

(c) For all $x, y \in \mathbb{R}$, $|2x + 3y| \leq 2\,|x| + 3\,|y|$.

[**TRUE**]   **FALSE**

(d) If $f : A \to B$ is uniformly continuous on A then it is still possible that there is a point $a \in A$ where f is discontinuous.

TRUE   [**FALSE**]

(e) Every Cauchy sequence of real numbers converges.

[**TRUE**]   **FALSE**

(f) Every bijective function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.

[**TRUE**]   **FALSE**

[9] **QUESTION 3.** For each of the sets E in the table below, answer ***YES*** or ***NO*** in each column to indicate whether or not E is open, dense in $\mathbb{R}$, or compact. *Do not justify your answers.*

Set E	**Open?**	**Dense in** $\mathbb{R}$**?**	**Compact?**
$\mathbb{R}$	**YES**	**YES**	**NO**
$\{3x + 2y : x, y \in \mathbb{R} \setminus \mathbb{Q}\}$	**NO**	**YES**	**NO**
$\{\sqrt{2}\} \cup \left\{\frac{\sqrt{2}}{n+1} : n \in \mathbb{N}\right\}$	**NO**	**NO**	**NO**

[6] **QUESTION 4.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the greatest lower bound ($\inf(E)$), the closure ($\overline{E}$), and the boundary (∂E). If the requested point or set does not exist, then indicate this with the symbol $\nexists$. *Do not justify your answers.*

E	$\inf(E)$	$\overline{E}$	∂E
$\mathbb{N}$	1	E	E
$\{\sqrt{2}\} \cup \left\{\frac{\sqrt{2}}{n+1} : n \in \mathbb{N}\right\}$	0	$E \cup \{0\}$	$E \cup \{0\}$

[10] **QUESTION 5.**

[3] (a) *Complete the formal definition:*
Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose x_0 is ___`an accumulation point of` E___. Then f is said to approach the limit L as x approaches x_0 if and only if

for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$, $x \neq x_0$, and $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Equivalently: $\forall \varepsilon > 0 \ \exists \delta > 0 \ \ni \ \big(x \in E \ \wedge \ 0 < |x - x_0| < \delta\big) \implies |f(x) - L| < \varepsilon.$

[7] (b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 1$. Use the formal definition to prove that $f(x)$ approaches 4 as x approaches 1.

Proof. Given $\varepsilon > 0$, we must find $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x - 1| < \delta$ then $|(3x + 1) - 4| < \varepsilon$.

Note that $|(3x + 1) - 4| = |3x - 3| = 3\,|x - 1|$.

Therefore, given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{3}$. Then, if $|x - 1| < \delta$,

$$|(3x + 1) - 4| = |3x - 3| = 3\,|x - 1| < 3\delta = 3\frac{\varepsilon}{3} = \varepsilon\,,$$

as required. □

[13] **QUESTION 6.**

[3] (a) (*Fill in the blanks.*) The ***completeness axiom*** for the set of real numbers states that if $E \subseteq \mathbb{R}$, $\underline{\quad E \neq \varnothing \quad}$ and $\underline{\text{ E is bounded }}$ then E has $\underline{\text{ a least upper bound }}$.

[3] (b) (*Fill in the blanks.*) Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers, and $\{s_n\}$ is another sequence of real numbers. The ***squeeze theorem*** for sequences states that if

(i) $s_n \leq x_n \leq t_n \quad \forall n \in \mathbb{N}$,

and (ii) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$

then $\{s_n\}$ converges and $\lim_{n\to\infty} x_n = L$.

[7] (c) Suppose that $E \subseteq \mathbb{R}$ and that E has a least upper bound ($\sup E = \alpha$). Prove that there is a sequence $\{e_n\}$ such that $e_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} e_n = \alpha$.

Hint: Consider two cases: $\alpha \in E$ or $\alpha \notin E$. In the latter case, for any $\varepsilon > 0$ there exists $x \in E$ such that $x > \alpha - \varepsilon$.

Let's consider the two cases, as suggested. (Note that in order to establish the result, it isn't actually necessary to consider the case $\alpha \in E$ separately. But that case is simpler, so it provided an opportunity to get somewhere without solving the full problem.)

Case $\alpha \in E$**:** In this case, just take $e_n = \alpha$ for all $n \in \mathbb{N}$. Then $e_n \to \alpha$ trivially.

Case $\alpha \notin E$**:** First, let's verify the statement in the hint. Suppose, in order to derive a contradiction, that the statement in the hint is false. Thus, there exists $\varepsilon > 0$ such that for all $x \in E$, $x \leq \alpha - \varepsilon$. But then $\alpha - \varepsilon$ is an upper bound for E that is less than $\alpha = \sup E$. $\Rightarrow\Leftarrow$. Therefore, the statement in the hint is true.

Since the statement in the hint is true, we can take advantage of it for $\varepsilon = \frac{1}{n}$, for any $n \in \mathbb{N}$. Thus for all $n \in \mathbb{N}$, there exists $e_n \in E$ such that $e_n > \alpha - \frac{1}{n}$. Moreover, since $e_n \in E$ we must have $e_n \leq \sup E = \alpha$. Thus,

$$\alpha - \frac{1}{n} < e_n \leq \alpha, \qquad \forall n \in \mathbb{N}. \tag{$*$}$$

We can now exploit the squeeze theorem: let $s_n = \alpha - \frac{1}{n}$ and $t_n = \alpha$ for all n. Then $s_n \to \alpha$ and $t_n \to \alpha$. Moreover, from (*) we have $s_n \leq e_n \leq t_n$ for all n. Hence, the Squeeze theorem implies that $e_n \to \alpha$. $\square$

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Note that question 7 is on the next page.

[14] **QUESTION 7.** A set $E \subseteq \mathbb{R}$ is ***compact*** if and only if it satisfies any of the following three equivalent properties. Complete the definition of each property:

[1] (a) E is closed and `bounded`;

[2] (b) E has the Bolzano-Weierstrass property, *i.e.,* every sequence in E ...

contains a subsequence that converges to a point in E.

[2] (c) E has the Heine-Borel property, *i.e.,* every open cover of E ...

contains a finite subcover of E.

[9] (d) Use one of the definitions above to prove that if A and B are both non-empty, compact subsets of $\mathbb{R}$ then $A \cup B$ is also compact.

Note: If you choose definition (a) then as part of your solution you **must prove** that the union of two closed sets is closed.

closed and bounded: Recall that E' refers to the set of accumulation points of a set E, and E is closed iff $E' \subseteq E$.

$A \cup B$ **is closed:** Let $x \in (A \cup B)'$. We must show that $x \in A \cup B$. If $x \in A'$ then $x \in A$ because A is closed; hence $x \in A \cup B$. Similarly, if $x \in B'$ then $x \in B$ because B is closed; hence $x \in A \cup B$. If $x \notin A'$ and $x \notin B'$ then there is a deleted neighbourhood of x that contains no points of A and no points of B, *i.e.,* no points of $A \cup B$, contradicting the fact that x is an accumulation point of $A \cup B$. Thus, either $x \in A'$ or $x \in B'$ (or both), which we have seen implies that $x \in A \cup B$.

$A \cup B$ **is bounded:** Since A is bounded, there exists $M_A > 0$ such that $\forall x \in A$, $|x| < M_A$. Similarly, $\exists M_B > 0 \ni \forall x \in B \; |x| < M_B$. Therefore, any $x \in A \cup B$ satisfies $|x| < M \equiv \max(M_A, M_B)$, *i.e.,* $A \cup B$ is bounded. $\square$

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Bolzano-Weierstrass : Let $\{x_n\}$ be a sequence in $A \cup B$. We must show that $\{x_n\}$ contains a subsequence that converges to a point in $A \cup B$. Since there are infinitely many points in $\{x_n\}$, there must be infinitely points of $\{x_n\}$ in at least one of A or B, *i.e.*, $\{x_n\}$ must contain a subsequence that is either strictly in A or strictly in B. Suppose $\{x_n\}$ has a subsequence $\{a_n\} \subseteq A$. Since A is compact, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a point $a \in A$. But $\{a_{n_k}\} \subseteq A \subseteq A \cup B$, so $\{a_{n_k}\}$ is a subsequence of $\{x_n\}$ that converges to a point $a \in A \subseteq A \cup B$. □

Heini-Borel : Let $\mathcal{U}$ be an open cover of $A \cup B$. We must show that $\mathcal{U}$ contains a finite subcover of $A \cup B$. Since $\mathcal{U}$ covers $A \cup B$, it certainly covers A, and since A is compact, $\mathcal{U}$ contains a finite subcover of A, say $\{U_1, \ldots, U_n\}$. Similarly, $\mathcal{U}$ covers B, so it contains a finite subcover of B, say $\{V_1, \ldots, V_m\}$. Therefore,

$$\{U_1, \ldots, U_n\} \bigcup \{V_1, \ldots, V_m\}$$

is a finite subcollection of $\mathcal{U}$ that covers $A \cup B$. □

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