# Mathematics 3A03 — Real Analysis I

TERM TEST #1 — 29 October 2019

**Duration:** 90 minutes

- Print your name and student number clearly in the space below, with one character in each box.

- Write your signature here: ________________________________________.

**Notes:**

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 10 **pages** (*i.e.,* **5 double-sided pages**). There are **7 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. There is one blank page after question 6 and an additional three blank pages at the end.
- The first 4 questions do not require any justification for your answers. For these, you will be assessed on your answers only. *Do not justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 60.

## GOOD LUCK and ENJOY!

MARKS

[2] **QUESTION 1.** (*Circle the correct answer.*) For each of the following sets, determine whether it is ***Countable*** or ***Uncountable***. *Do not justify your answers.*

(a) $\mathbb{N} \cap \mathbb{R}$ **Countable** **Uncountable**

(b) $\left\{2^{k/2}k^n : n \in \mathbb{N},\ k \in \mathbb{Z}\right\}$ **Countable** **Uncountable**

[6] **QUESTION 2.** (*Circle the correct answer.*) Determine whether each of the following statements is **TRUE** or **FALSE**. *Do not justify your answers.*

(a) If $A \subseteq \mathbb{Q}$ is bounded and $A \neq \varnothing$ then $A$ has a least upper bound that is a rational number.

**TRUE** **FALSE**

(b) Every non-empty subset of $\mathbb{N}$ is bounded below.

**TRUE** **FALSE**

(c) For all $x, y \in \mathbb{R}$, $|2x + 3y| \leq 2\,|x| + 3\,|y|$.

**TRUE** **FALSE**

(d) If $f : A \to B$ is uniformly continuous on $A$ then it is still possible that there is a point $a \in A$ where $f$ is discontinuous.

**TRUE** **FALSE**

(e) Every Cauchy sequence of real numbers converges.

**TRUE** **FALSE**

(f) Every bijective function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.

**TRUE** **FALSE**

[9] **QUESTION 3.** For each of the sets $E$ in the table below, answer ***YES*** or ***NO*** in each column to indicate whether or not $E$ is open, dense in $\mathbb{R}$, or compact. *Do not justify your answers.*

| **Set** $E$ | **Open?** | **Dense in** $\mathbb{R}$**?** | **Compact?** |
|---|---|---|---|
| $\mathbb{R}$ | | | |
| $\{3x + 2y : x, y \in \mathbb{R} \setminus \mathbb{Q}\}$ | | | |
| $\{\sqrt{2}\} \cup \left\{\frac{\sqrt{2}}{n+1} : n \in \mathbb{N}\right\}$ | | | |

[6] **QUESTION 4.** For each of the sets $E$ in the table below, fill in the associated point or set in each column, *i.e.*, for each set $E$ state the greatest lower bound ($\inf(E)$), the closure ($\overline{E}$), and the boundary ($\partial E$). If the requested point or set does not exist, then indicate this with the symbol $\nexists$. *Do not justify your answers.*

| $E$ | $\inf(E)$ | $\overline{E}$ | $\partial E$ |
|---|---|---|---|
| $\mathbb{N}$ | | | |
| $\{\sqrt{2}\} \cup \left\{\frac{\sqrt{2}}{n+1} : n \in \mathbb{N}\right\}$ | | | |

[10] **QUESTION 5.**

[3] (a) *Complete the formal definition:*
Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose $x_0$ is ____________________________________. Then $f$ is said to approach the limit $L$ as $x$ approaches $x_0$ if and only if

[7] (b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 1$. Use the formal definition to prove that $f(x)$ approaches 4 as $x$ approaches 1.

[13] **QUESTION 6.**

[3] (a) (*Fill in the blanks.*) The ***completeness axiom*** for the set of real numbers states that if $E \subseteq \mathbb{R}$, _______________ and _______________ then $E$ has _________________________.

[3] (b) (*Fill in the blanks.*) Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers, and $\{s_n\}$ is another sequence of real numbers. The ***squeeze theorem*** for sequences states that if

(i) ____________________________________________________________

and (ii) ____________________________________________________________

then $\{s_n\}$ converges and ______________________________.

[7] (c) Suppose that $E \subseteq \mathbb{R}$ and that $E$ has a least upper bound ($\sup E = \alpha$). Prove that there is a sequence $\{e_n\}$ such that $e_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} e_n = \alpha$.

*Hint:* Consider two cases: $\alpha \in E$ or $\alpha \notin E$. In the latter case, for any $\varepsilon > 0$ there exists $x \in E$ such that $x > \alpha - \varepsilon$.

*This page has been left blank intentionally to provide extra space for question 6 if needed.*
***Note that question 7 is on the next page.***

[14] **QUESTION 7.** A set $E \subseteq \mathbb{R}$ is ***compact*** if and only if it satisfies any of the following three equivalent properties. Complete the definition of each property:

[1] (a) $E$ is closed and ____________________;

[2] (b) $E$ has the Bolzano-Weierstrass property, *i.e.*, every sequence in $E$ ...

[2] (c) $E$ has the Heine-Borel property, *i.e.*, every open cover of $E$ ...

[9] (d) Use one of the definitions above to prove that if $A$ and $B$ are both non-empty, compact subsets of $\mathbb{R}$ then $A \cup B$ is also compact.

*Note:* If you choose definition (a) then as part of your solution you **must prove** that the union of two closed sets is closed.

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