Student Name: Harry Potter Student Number: 33550336

Mathematics 3A03 — Real Analysis I

TERM TEST #2 — 27 November 2017

SOLUTIONS

Duration: 90 minutes

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 6 **pages** and includes **7 questions**. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. Use the backs of pages if you need more space. The final page is blank to provide extra space for the final question.
- The first 4 questions are *multiple choice* or *fill in the blank*. *Do not justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

Question	Mark
1	1
2	5
3	5
4	7
Subtotal	18

Question	Mark
5	10
6	10
7	12
Subtotal	32

Total	50

Name of Instructor: Isaac Newton **Name of TA:** Albert Einstein

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GOOD LUCK and ENJOY!

MARKS

[1] **QUESTION 1.** *In order to obtain any credit for this question, both parts must be answered in clear handwriting in the location(s) specified.*

(a) What are the names of the instructor and TA whose sections you attend?
Answer at the bottom of the front page only.

(b) What are your name and student number?
Answer at the top of every page of this test.

[5] **QUESTION 2.** (*Circle each correct answer.*) A set $E \subseteq \mathbb{R}$ is ***compact*** if and only if

(a) $\boxed{\checkmark}$ E is bounded and contains all its accumulation points;

(b) $\mathbb{R} \setminus E$ is open and unbounded;

(c) $\mathbb{R} \setminus E$ is compact;

(d) every sequence of points chosen from E has a subsequence that converges;

(e) $\boxed{\checkmark}$ every open cover of E has a finite subcover.

[5] **QUESTION 3.** (*Circle each correct answer.*) Suppose $a < b$. If f is continuous on $[a, b]$ and $c \in (a, b)$ then

(a) $\boxed{\checkmark}$ f is bounded on (a, c);

(b) $f(c) = 0$;

(c) $\frac{f(b)-f(a)}{b-a} = f'(c)$;

(d) $\boxed{\checkmark}$ $f([a, c])$ is compact;

(e) $\boxed{\checkmark}$ f is uniformly continuous on (a, c).

[7] **QUESTION 4.** Suppose A and B are non-empty subsets of $\mathbb{R}$ and $f : A \to B$ is a function. *In each of the following, fill in the blank with one of* $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ *or* `NI`, *where* `NI` *means "no implication"*, i.e., *neither statement necessarily implies the other.*

(a) A, B finite ___$\Longrightarrow$___ $A \cap B$ finite.

(b) A, B compact ___$\Longrightarrow$___ $A \cup B$ compact.

(c) A compact ___`NI`___ $f(A)$ compact.

(d) f continuous on A ___`NI`___ f achieves a max and a min on A.

For the following, assume $A = [a, b]$ with $a < b$.

(e) f continuous on A ___$\Longleftarrow$___ f differentiable on A.

(f) f integrable on A ___$\Longleftarrow$___ f continuous on A.

(g) f has intermediate value property on A ___$\Longleftarrow$___ f differentiable on A.

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[10] **QUESTION 5.** Let f be integrable on $[a,b]$, and define F on $[a,b]$ by

$$F(x) = \int_a^x f\,.$$

(a) The First Fundamental Theorem of Calculus (FFTC) states that if f is continuous at $c \in [a,b]$, then F is differentiable at c, and $F'(c) =$ $f(c)$.

Suppose that $c \in (a,b)$. For each of the following statements, state whether the proposition is TRUE or FALSE, and support your claim with either a (short) proof or a counterexample.

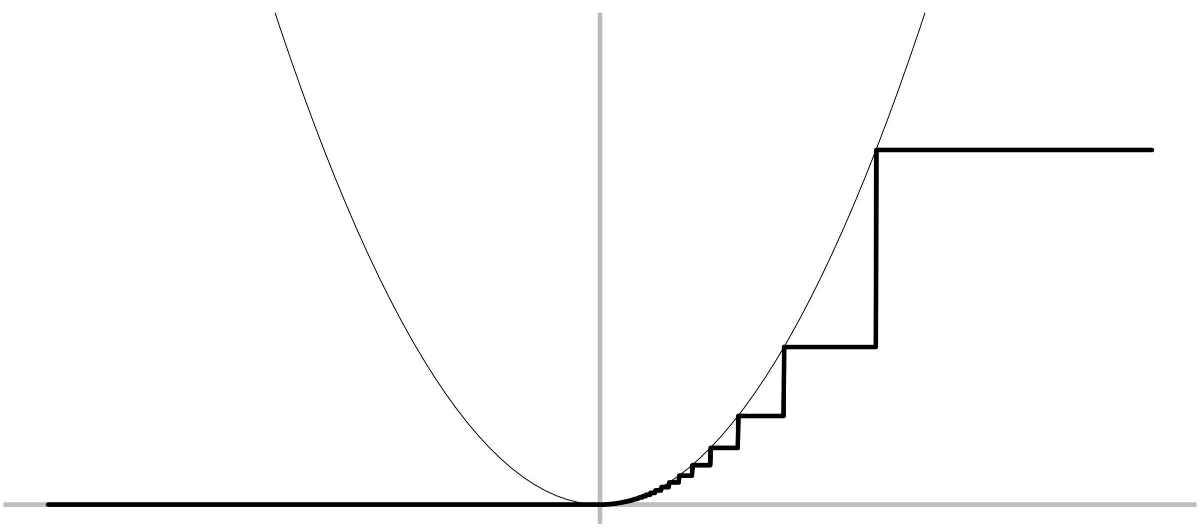
(b) $\boxed{\texttt{TRUE}}$ FALSE *If f is differentiable at c, then F is differentiable at c.*

Solution: If f is differentiable at c then it is continuous at c. Hence by FFTC, F is differentiable at c (and $F'(c) = f(c)$). □

(c) TRUE $\boxed{\texttt{FALSE}}$ *If f is differentiable at c, then F' is continuous at c.*

Solution: Define $f : [-1,1] \to \mathbb{R}$ via

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{(n+1)^2} & \text{if } \frac{1}{n+1} < x \le \frac{1}{n}, \end{cases} \quad n \in \mathbb{N}.$$



f is "trapped" between 0 and x^2, so f is differentiable at 0, and hence by part (a) it follows that F is differentiable at 0 and $F'(0) = f(0) = 0$. In addition, provided $x \neq 1/n$ for $n \in \mathbb{N}$, f is continuous at x. So F is differentiable at $x \neq 1/n$ and $F'(x) = f(x)$. Now consider the sequence $x_n = \frac{1}{n-1}$ for $1 < n \in \mathbb{N}$. $F'_-(x_n) = \frac{1}{(n+1)^2} < \frac{1}{n^2} = F'_+(x_n)$ so F is <u>not</u> differentiable at the sequence of points $\{x_n\}$. Thus F' does not even exist at a sequence of points that approach 0, so F' certainly cannot be continuous at 0. □

(d) $\boxed{\texttt{TRUE}}$ FALSE *If f' is differentiable at c, then F' is continuous at c.*

Solution: If f' is differentiable at c, then f' must be defined in a neighbourhood of c, *i.e.*, f must be differentiable in a neighbourhood of c. Hence f must be continuous in this neighbourhood, say $\mathcal{N}$, of c. At each $x \in \mathcal{N}$, f is continuous so F is differentiable and $F'(x) = f(x)$. But f is differentiable in $\mathcal{N}$, hence so is F'. $\square$

(e) TRUE $\boxed{\texttt{FALSE}}$ *If F is differentiable on (a, b), then f satisfies the intermediate value property on (a, b).*

Solution: Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ if $x \neq 1/2$ and $f(1/2) = 1$. Then $F \equiv 0$ so F is certainly differentiable on $(0, 1)$, but f does not satisfy the IVP. $\square$

[10] **QUESTION 6.**

(a) For $a < b$ and $c \in (a, b)$, state the formal ε-δ definition of "the function $h : (a, b) \to \mathbb{R}$ is ***continuous*** at c".

Solution: $\forall \varepsilon > 0 \; \exists \delta > 0$ such that if $x \in (a, b)$ and $|x - c| < \delta$ then $|h(x) - h(c)| < \varepsilon$.

(b) Let $I = (a, b)$ and suppose the two functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous at every $x \in I$. If

$$h(x) = f(x) + 2\,g(x) \quad \text{for all} \quad x \in I,$$

prove directly from the formal definition that h is continuous at every $x \in I$.

Solution: Given $x \in I$ and $\varepsilon > 0$, choose $\delta_1 > 0$ (depending on both x and ε) such that if $y \in I$ and $|y - x| < \delta_1$ then$|f(y) - f(x)| < \frac{\varepsilon}{2}$. Also choose $\delta_2 > 0$ such that if $y \in I$ and $|y - x| < \delta_2$ then $|g(y) - g(x)| < \frac{\varepsilon}{4}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for any $y \in I$ with $|y - x| < \delta$, it follows that

$$|f(y) - f(x)| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(y) - g(x)| < \frac{\varepsilon}{4}\,.$$

We then have

$$\begin{aligned}
|h(y) - h(x)| &= \big|\big(f(y) + 2\,g(y)\big) - \big(f(x) + 2\,g(x)\big)\big| \\
&= \big|\big(f(y) - f(x)\big) + \big(2\,g(y) - 2\,g(x)\big)\big| \\
&= \big|\big(f(y) - f(x)\big) + 2\big(g(y) - g(x)\big)\big| \\
&\leq |f(y) - f(x)| + 2\,|g(y) - g(x)| \qquad \text{(triangle inequality)} \\
&< \frac{\varepsilon}{2} + 2 \cdot \frac{\varepsilon}{4} \;=\; \varepsilon\,.
\end{aligned}$$

Thus, h is continuous at x. Since x was an arbitrary point in I, it follows that h is continuous at every $x \in I$. $\square$

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[12] **QUESTION 7.** Let $a < b$ and suppose f is a nondecreasing function defined on $[a, b]$.

(a) Prove that f is bounded on $[a, b]$.

Solution: Since f is nondecreasing on $[a, b]$, we have $f(x) \geq f(a)$ $\forall x \in [a, b]$ and $f(x) \leq f(b)$ $\forall x \in [a, b]$. Thus

$$f(a) \leq f(x) \leq f(b)\,, \qquad \text{for all } x \in [a, b],$$

i.e., $f(a)$ is a lower bound for f on $[a, b]$ and $f(b)$ is an upper bound for f on $[a, b]$. Hence f is bounded on $[a, b]$. □

(b) Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Give the definition of the upper sum $U(f, P)$ and of the lower sum $L(f, P)$.

Solution: Let

$$m_i = \inf \{\, f(x) \;:\; x \in [t_{i-1}, t_i] \,\}\,,$$
$$M_i = \sup \{\, f(x) \;:\; x \in [t_{i-1}, t_i] \,\}\,.$$

The ***lower sum*** of f for P, denoted by $L(f, P)$, is

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})\,.$$

The ***upper sum*** of f for P, denoted by $U(f, P)$, is

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})\,.$$

(c) Prove that f is integrable. *Hint: Suppose that $t_i - t_{i-1} = \delta$ for each i. Prove that*

$$U(f, P) - L(f, P) \leq \delta[f(b) - f(a)]$$

and use this to show that f is integrable.

Solution: We know from part (a) that f is bounded on $[a, b]$. It remains to show that given any $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Consider an evenly spaced partition P with $t_i - t_{i-1} = \delta$ for $i = 1, \ldots, n$. We then have

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} m_i\delta = \delta \sum_{i=1}^{n} m_i\,,$$
$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = \sum_{i=1}^{n} M_i\delta = \delta \sum_{i=1}^{n} M_i\,,$$

and hence

$$U(f,P) - L(f,P) = \delta \sum_{i=1}^{n} (M_i - m_i)\,.$$

Since f is nondecreasing on $[a,b]$, on each subinterval $[t_{i-1}, t_i]$ we have $m_i = f(t_{i-1})$ and $M_i = f(t_i)$. Therefore

$$\begin{aligned}
U(f,P) - L(f,P) &= \delta \sum_{i=1}^{n} \big[f(t_i) - f(t_{i-1})\big] \\
&= \delta\Big[f(t_1) - f(t_0) + f(t_2) - f(t_1) + \cdots + f(t_n) - f(t_{n-1})\Big] \\
&= \delta\big[f(t_n) - f(t_0)\big] \\
&= \delta\big[f(b) - f(a)\big]\,.
\end{aligned}$$

There are now two cases to consider.

(i) $f(b) = f(a)$. This occurs if f is, in fact, a constant function. In this case, given $\varepsilon > 0$ choose any evenly spaced partition (*e.g.*, $P = \{a, b\}$ will do) and note from above that $U(f,P) - L(f,P) = 0 < \varepsilon$.

(ii) $f(b) > f(a)$. Given $\varepsilon > 0$, choose an evenly spaced partition P as above, but more specifically with

$$\delta = \frac{\varepsilon}{2[f(b) - f(a)]}\,.$$

Then

$$\begin{aligned}
U(f,P) - L(f,P) &= \delta\big[f(b) - f(a)\big] \\
&= \frac{\varepsilon}{2[f(b) - f(a)]}\big[f(b) - f(a)\big] \\
&= \frac{\varepsilon}{2} \; < \; \varepsilon\,,
\end{aligned}$$

as required. $\square$

THE END