Student Name: _____   Student Number: _____

Mathematics 3A03 — Real Analysis I

TERM TEST #2 — 27 November 2017

Duration: 90 minutes

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of **6 pages** and includes **7 questions**. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. Use the backs of pages if you need more space. The final page is blank to provide extra space for the final question.
- The first 4 questions are *multiple choice* or *fill in the blank*. Do *not* justify your answers to these questions.
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

Question	Mark
1	
2	
3	
4	
Subtotal	

Question	Mark
5	
6	
7	
Subtotal	

Total	

Name of Instructor: _____   **Name of TA:** _____

GOOD LUCK and ENJOY!

MARKS

[1] **QUESTION 1.** *In order to obtain any credit for this question, both parts must be answered in <u>clear handwriting</u> in the location(s) specified.*

   (a) What are the names of the instructor and TA whose sections you attend?
       Answer at the bottom of the <u>front page</u> only.

   (b) What are your name and student number?
       Answer at the top of <u>every page</u> of this test.

[5] **QUESTION 2.** (*Circle each correct answer.*) A set $E \subseteq \mathbb{R}$ is **compact** if and only if

   (a) E is bounded and contains all its accumulation points;

   (b) $\mathbb{R} \setminus E$ is open and unbounded;

   (c) $\mathbb{R} \setminus E$ is compact;

   (d) every sequence of points chosen from E has a subsequence that converges;

   (e) every open cover of E has a finite subcover.

[5] **QUESTION 3.** (*Circle each correct answer.*) Suppose $a < b$. If f is continuous on $[a, b]$ and $c \in (a, b)$ then

   (a) f is bounded on (a, c);

   (b) $f(c) = 0$;

   (c) $\frac{f(b)-f(a)}{b-a} = f'(c)$;

   (d) $f([a, c])$ is compact;

   (e) f is uniformly continuous on (a, c).

[7] **QUESTION 4.** Suppose A and B are non-empty subsets of $\mathbb{R}$ and $f : A \to B$ is a function. *In each of the following, fill in the blank with one of* $\implies$, $\impliedby$, $\iff$ *or* NI, *where* NI *means "no implication",* i.e., *neither statement necessarily implies the other.*

   (a) A, B finite _____ $A \cap B$ finite.

   (b) A, B compact _____ $A \cup B$ compact.

   (c) A compact _____ $f(A)$ compact.

   (d) f continuous on A _____ f achieves a max and a min on A.

For the following, assume $A = [a, b]$ with $a < b$.

   (e) f continuous on A _____ f differentiable on A.

   (f) f integrable on A _____ f continuous on A.

   (g) f has intermediate value property on A _____ f differentiable on A.

[10] **QUESTION 5.** Let f be integrable on $[a, b]$, and define F on $[a, b]$ by

$$F(x) = \int_a^x f \, .$$

(a) The First Fundamental Theorem of Calculus (FFTC) states that if f is _____
at $c \in [a, b]$, then F is _____ at c, and $F'(c) = $ _____ .

Suppose that $c \in (a, b)$. For each of the following statements, state whether the proposition
is TRUE or FALSE, and support your claim with either a (short) proof or a counterexample.

(b) TRUE   FALSE   *If f is differentiable at c, then F is differentiable at c.*

(c) TRUE   FALSE   *If f is differentiable at c, then F' is continuous at c.*

(d) TRUE   FALSE   *If f' is differentiable at c, then F' is continuous at c.*

(e) TRUE   FALSE   *If F is differentiable on (a, b), then f satisfies the intermediate value*
property on (a, b).

[10] **QUESTION 6.**

(a) For $a < b$ and $c \in (a, b)$, state the formal ε-δ definition of "the function $h : (a, b) \to \mathbb{R}$ is **continuous** at c".

(b) Let $I = (a, b)$ and suppose the two functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous at every $x \in I$. If

$$h(x) = f(x) + 2\,g(x) \quad \text{for all} \quad x \in I,$$

prove directly from the formal definition that h is continuous at every $x \in I$.

[12] **QUESTION 7.**   Let $a < b$ and suppose f is a nondecreasing function defined on $[a, b]$.

(a) Prove that f is bounded on $[a, b]$.

(b) Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$. Give the definition of the upper sum $U(f, P)$ and of the lower sum $L(f, P)$.

(c) Prove that f is integrable. *Hint: Suppose that $t_i - t_{i-1} = \delta$ for each i. Prove that*

$$U(f, P) - L(f, P) \le \delta[f(b) - f(a)]$$

and use this to show that f is integrable.

THE END