Student Name: ____________________ Student Number: _______________

Mathematics 3A03 — Real Analysis I

TERM TEST #1 (Solutions)— 23 October 2017

Duration of Standard Test: 90 minutes

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 8 **pages** (*i.e.*, **4 double-sided pages**). There are **9 questions** in total. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. The final page is blank to provide extra space if needed.
- The first 6 questions do not require any justification for your answers. For these, you will be assessed on your answers only. *Do not justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

Question	Mark
1	
2	
3	
4	
5	
Subtotal	

Question	Mark
6	
7	
8	
9	
Subtotal	

Total	

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(Answers in Blue)

MARKS

[1] **QUESTION 1.** *In order to obtain any credit for this question, both parts must be answered in clear handwriting at the top of every page of this test.*

(a) What is your name?

(b) What is your student number?

[5] **QUESTION 2.** (*Circle the correct answer.*) For each of the following sets, determine whether it is ***Countable*** or ***Uncountable***. *Do not justify your answers.*

(a) The interval $(2, 4)$

Countable **Uncountable**

(b) $\mathbb{Z} \times \mathbb{N}$

Countable **Uncountable**

(c) $\mathbb{R} \times \mathbb{R}$

Countable **Uncountable**

(d) $\{x \in (0, 1) \ : \ x \notin \mathbb{Q}\}$

Countable **Uncountable**

(e) $\{m + n\pi : m, n \in \mathbb{Q}\}$

Countable **Uncountable**

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[4] **QUESTION 3.** (*Circle the correct answer.*) Determine whether each the following statements is ***True*** or ***False***. *Do not justify your answers.*

(a) Every non-empty subset of $\mathbb{N}$ has a least element.

True **False**

(b) There are Cauchy sequences of real numbers that do not converge.

True **False**

(c) For every $x > 0$, there is a positive rational number $q \in \mathbb{Q}$ so that $q < x$.

True **False**

(d) Every monotone sequence converges.

True **False**

[4] **QUESTION 4.** (*Circle the correct answer.*) Determine whether each the following statements is ***Never True***, ***Sometimes True***, or ***Always True***. *Do not justify your answers.*

(a) If $\{x_n\}$ and $\{y_n\}$ are both convergent sequences, and $y_n \neq 0$ for all n, then $\{x_n/y_n\}$ is a convergent sequence.

Never True **Sometimes True** **Always True**

(b) If $A_1, A_2, \ldots$ are open subsets of $\mathbb{R}$, then the intersection $\cap_{n\in\mathbb{N}} A_n$ is open.

Never True **Sometimes True** **Always True**

(c) Suppose $E \subset \mathbb{R}$. Then the closure of the interior of E is equal to E.

Never True **Sometimes True** **Always True**

(d) Suppose $\{x_n\}$ is a bounded, decreasing sequence. Then $\{x_n\}$ has exactly one convergent subsequence.

Never True **Sometimes True** **Always True**

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[6] **QUESTION 5.** For each of the sets E in the table below, answer ***YES*** or ***NO*** in each column to indicate whether or not E is open, bounded or dense in $\mathbb{R}$. *Do not justify your answers.*

Set E	**Open?**	**Bounded?**	**Dense in** $\mathbb{R}$**?**
$(\sqrt{2}, 3+\sqrt{2}]$	NO	YES	NO
$\{m + n\pi : m, n \in \mathbb{Q}\}$	NO	NO	YES

[6] **QUESTION 6.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the greatest lower bound ($\inf(E)$), the closure ($\overline{E}$), and the set of accumultation points (E'). If the requested point or set does not exist, then indicate this with the symbol $\nexists$. *Do not justify your answers.*

Set E	$\inf(E)$	$\overline{E}$	E'
$(\sqrt{2}, 3+\sqrt{2}) \cap \mathbb{Q}$	$\sqrt{2}$	$[\sqrt{2}, 3+\sqrt{2}]$	$[\sqrt{2}, 3+\sqrt{2}]$
$\left\{\frac{n+1}{n} : n \in \mathbb{N}\right\}$	1	$E \cup \{1\}$	1

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[12] **QUESTION 7.**

(a) Let $\{s_n\}$ be a sequence. State the formal definition of "the sequence $\{s_n\}$ converges as $n \to \infty$".

Answer 1: The sequence $\{s_n\}$ converges as $n \to \infty$ if there exists some $L \in \mathbb{R}$ so that, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, for all $n \in \mathbb{N}$ with $n \geq N$,

$$|s_n - L| < \varepsilon.$$

Answer 2: The sequence $\{s_n\}$ converges as $n \to \infty$ if there exists some $L \in \mathbb{R}$ so that, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that if $n \geq N$ is an integer, then

$$|s_n - L| < \varepsilon.$$

(b) Suppose $\{x_n\}$ is a bounded sequence. Use the formal definition to prove that the sequence $\left\{ \dfrac{x_n}{n^2+1} \right\}$ converges as $n \to \infty$.

Proof: Take $L = 0$ and let $\varepsilon > 0$. Since $\{x_n\}$ is bounded, there is some M so that $|x_n| \leq M$ for all $n \in \mathbb{N}$. By increasing M, if necessary, we may assume $M > 0$ and $M/\varepsilon - 1 > 0$. Pick $N \in \mathbb{N}$ so that

$$N > \sqrt{M/\varepsilon - 1}.$$

If $n \geq N$ is an integer, then

$$\begin{aligned} \left| \tfrac{x_n}{n^2+1} - L \right| &\leq \tfrac{M}{n^2+1} \\ &\leq \tfrac{M}{N^2+1} \\ &< \tfrac{M}{M/\varepsilon - 1 + 1} \\ &= \varepsilon \end{aligned}$$

□

[6] **QUESTION 8.** Show that if $S \subseteq \mathbb{R}$ is open and non-empty, then S is uncountable. (You may use, without proof, the fact that the interval $(0,1)$ is uncountable.)

Proof: Since S is non-empty, there is some $x \in S$. Since S is open, there is some $(a,b) \subset S$ with $x \in (a,b)$. In class, it was shown that if a set is countable, then any subset is also countable. Consequently, to prove that S is uncountable, it suffices to show that (a,b) is uncountable. To see this, we will construct a bijection between (a,b) and $(0,1)$; that the latter is uncountable implies that the former is as well.

A bijection is given by the function $f : (0,1) \to (a,b)$ defined by

$$f(x) = bx + (1-x)a = (b-a)x + a.$$

This is well-defined because if $x \in (0,1)$, then $0 < x < 1$, and so

$$a < (b-a)x + a < b.$$

Hence $f(x) \in (a,b)$. This is injective because if $f(x) = f(y)$, then $(b-a)x = (b-a)y$, so $x = y$ (note that $b - a \neq 0$ because the interval (a,b) is non-empty). This is surjective because if $c \in (a,b)$, then $(b-a)^{-1}(c-a) \in (0,1)$ and $f((b-a)^{-1}(c-a)) = c$. □

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[6] **QUESTION 9.** Suppose $S \subseteq \mathbb{R}$ is a closed, bounded, and non-empty set of real numbers. Show that S contains its supremum: $\sup(S) \in S$.

Proof 1: Since S is bounded and non-empty, we know that $\sup(S)$ exists. Since S is closed, we have $S = \overline{S}$, so it suffices to show that $\sup(S) \in \overline{S}$. One definition of $\overline{S}$ is as the set of points $x \in \mathbb{R}$ so that, for all $\varepsilon > 0$, the intersection $(x-\varepsilon, x+\varepsilon) \cap S$ is non-empty. We will show this holds for $x = \sup(S)$. Let $\varepsilon > 0$. Then $\sup(S) - \varepsilon < \sup(S)$, so $\sup(S) - \varepsilon$ is not an upper bound for S. This implies there is some $y \in S$ so that

$$y > \sup(S) - \varepsilon.$$

On the other hand, $\sup(S)$ is an upper bound, so $y \leq \sup(S)$. This implies $y \in (x - \varepsilon, x + \varepsilon) \cap S$, and hence this intersection is non-empty. □

Proof 2: Since S is bounded and non-empty, we know that $\sup(S)$ exists. For sake of contradiction, assume S is closed, but $\sup(S) \notin S$. Let $\varepsilon > 0$. Then since $\sup(S)$ is the least upper bound and $\sup(S) - \varepsilon < \sup(S)$, it follows that $\sup(S) - \varepsilon$ is not an upper bound for S. This implies there is some point $x \in S$ between $\sup(S) - \varepsilon$ and $\sup(S)$; this point cannot be $\sup(S)$ because we have assumed $\sup(S)$ is not in S. Then we have that

$$S \backslash \{\sup(S)\} \cap (\sup(S) - \varepsilon, \sup(S) + \varepsilon) \neq \varnothing$$

is non-empty. This is true for all $\varepsilon > 0$, so it follows that $\sup(S)$ is an accumulation point for S. Since S is closed, it contains all of its accumulation points, and so $\sup(S) \in S$, which is a contradiction. □

Proof 3: Since S is bounded and non-empty, we know that $\sup(S)$ exists. For sake of contradiction, assume S is closed, but $\sup(S) \notin S$. Then $\sup(S)$ lies in the complement S^c. Since S is closed, the complement S^c is open. This implies that there is some $\varepsilon > 0$ so that $(\sup(S) - \varepsilon, \sup(S) + \varepsilon) \subset S^c$. Since $\sup(S)$ is an upper bound for S, all points of S are no greater than $\sup(S)$. They cannot lie in the region $(\sup(S) - \varepsilon, \sup(S))$ either, since this would contradict the fact that this set lies in S^c. Consequently, $x \leq \sup(S) - \varepsilon$ for all $x \in S$, so $\sup(S) - \varepsilon$ is an upper bound. Since $\sup(S) - \varepsilon < \sup(S)$, this contradicts the assumption that $\sup(S)$ is a least upper bound. □

Proof 4: Since $S \neq \varnothing$ and S is bounded, S has a least upper bound, say $L = \sup(S)$. Given any $\varepsilon > 0$ we can find $x \in S$ such that $x > L - \varepsilon$, *i.e.*, $L - x < \varepsilon$ (otherwise $L - \varepsilon$ would be an upper bound of S that is less than L). Consequently, for each $n \in \mathbb{N}$, we can find $x_n \in S$ such that $L - x_n < \frac{1}{n}$, *i.e.*, there is a sequence $\{x_n\}$ in S that converges to L. If $\{x_n\}$ is eventually constant then that constant is L (since $L - c < \varepsilon$ for all $\varepsilon > 0$ implies $c = L$) so $L \in S$. Otherwise, L is an accumulation point of S; but S is closed, so $L \in S$. □

THE END