Student Name: Harry Potter   Student Number: 33550336

Mathematics 3A03 — Real Analysis I — Prof. David Earn

TERM TEST #2 — 30 November 2016

SOLUTIONS

Duration: 50 minutes

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of **7 pages** and includes **8 questions**. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. Use the backs of pages if you need more space. The final page is blank to provide extra space for the final question.
- The first 4 questions are *multiple choice*. *Do not justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

Question	Mark
1	1
2	3
3	3
4	3
Subtotal	10

Question	Mark
5	6
6	4
7	15
8	15
Subtotal	40

Total	50

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GOOD LUCK and ENJOY!

MARKS

[1] **QUESTION 1.** *In order to obtain any credit for this question, both parts must be answered in clear handwriting at the top of every page of this test.*

(a) What is your name?

(b) What is your student number?

[3] **QUESTION 2.** (*Circle each correct answer.*) Suppose $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $a \in A$. Then f is ***continuous*** at a if and only if

(a) $\lim_{x \to a} f(x) = f(a)$.

(b) ✓ either a is an isolated point of A or a is an accumulation point of A and $\lim_{x \to a} f(x) = f(a)$.

(c) $\lim_{x \to x_0} f(x) = f(x_0)$ for all $x_0 \in A$.

(d) ✓ for any sequence $\{x_n\}$ in A, if $x_n \to a$ then $f(x_n) \to f(a)$.

(e) ✓ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in A$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

[3] **QUESTION 3.** (*Circle each correct answer.*) If f is continuous on $[a, b]$ then

(a) ✓ f is uniformly continuous on $[a, b]$;

(b) ✓ f is bounded on $[a, b]$;

(c) ✓ there exists $x_0 \in [a, b]$ such that $f(x_0) = \sup\{f(x) : x \in [a, b]\}$;

(d) f is differentiable on $[a, b]$;

(e) ✓ f is integrable on $[a, b]$.

[3] **QUESTION 4.** (*Circle each correct answer.*) A set $E \subseteq \mathbb{R}$ is ***compact*** if and only if

(a) ✓ E is bounded and contains all its accumulation points;

(b) every sequence of points chosen from E has a subsequence that converges;

(c) ✓ every open cover of E can be reduced to a finite subcover;

(d) $\mathbb{R} \setminus E$ is compact;

(e) $\mathbb{R} \setminus E$ is open and unbounded.

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[6] **QUESTION 5.** For each of the functions f in the table below, answer YES or NO in each column to indicate whether or not f has the indicated property. *Do not justify your answers.*

Function f	(a) f is differentiable on $(-1,1)$	(b) f is integrable on $[-1,1]$
$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in \mathbb{R},\ x \neq 0 \end{cases}$	**NO**	**YES**
$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$	**NO**	**NO**
$f(x) = \begin{cases} 0 & x = 1 \\ \frac{1}{1-x} & x \in \mathbb{R},\ x \neq 1 \end{cases}$	**YES**	**NO**

[4] **QUESTION 6.** For each of the functions f in the table below, answer YES or NO in each column to indicate whether or not f has the indicated property. *Do not justify your answers.*

Function f	(a) f is a derivative *i.e.*, $\exists g$ such that $f = g'$	(b) f is an integral *i.e.*, $\exists g$ such that $f(x) = \int_0^x g(x)\,dx$
$f(x) = 0 \quad \forall x \in \mathbb{R}$	**YES** ($g \equiv 0$)	**YES** ($g \equiv 0$)
$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in \mathbb{R},\ x \neq 0 \end{cases}$	**NO** (Darboux's theorem)	**NO** (integrals are continuous)

Note: The justifications given in this table were not required.

[15] **QUESTION 7.**

(a) State the formal definition of "the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x".

Solution:

f is differentiable at x iff $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists.

Equivalently, f is differentiable at x iff $\lim_{y\to x} \frac{f(y)-f(x)}{y-x}$ exists.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with the property that

$$|f(x) - f(y)| \le |x-y|^2 , \qquad \text{for all } x, y \in \mathbb{R}.$$

Prove that f is differentiable at every $x \in \mathbb{R}$, and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Solution:

Choose any $x \in \mathbb{R}$. Then, for any $y \neq x$ we have

$$\left| \frac{f(y)-f(x)}{y-x} \right| \le |y-x| ,$$

i.e.,

$$-|y-x| \le \frac{f(y)-f(x)}{y-x} \le |y-x| .$$

Since $\lim_{y\to x} |y-x| = 0$, the squeeze theorem implies that $\lim y \to x$ of the quantity in the middle exists and is equal to 0, *i.e.*, $f'(x)$ exists and $f'(x) = 0$. □

(c) For f as in part (b), prove that f must be a constant function.
<u>*Hint*</u>: Prove that $f(x) = f(0)$ for all $x \in \mathbb{R}$.

Solution:

If $x > 0$ then apply the *Mean Value Theorem* to the interval $[0, x]$ to infer $\exists \xi \in (0, x)$ such that

$$f(x) - f(0) = f'(\xi)(x-0) .$$

But $f'(x) = 0$ for all $x \in \mathbb{R}$, so in particular $f'(\xi) = 0$. Therefore, $f(x) - f(0) = 0$, *i.e.*, $f(x) = f(0)$. For $x < 0$, consider the interval $[x, 0]$ instead. □

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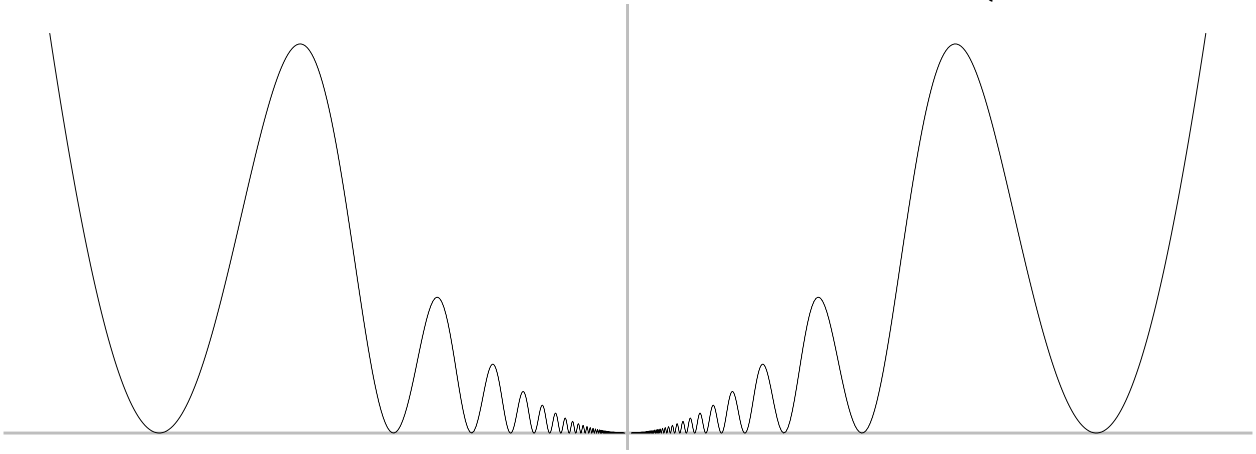
[15] **QUESTION 8.** Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \left(x \sin \frac{1}{x}\right)^2, & x \neq 0, \\ 0, & x = 0. \end{cases}$



(a) How do you know that f is integrable on $[0, 1]$?

Solution:

The function $\left(x \sin \frac{1}{x}\right)^2$ is well-defined and continuous for all $x \neq 0$. Moreover,

$$\lim_{x \to 0} f(x) = 0 = f(0)\,,$$

so f is also continuous at 0. Any continuous function on a closed interval is integrable, so f is integrable on $[0, 1]$.

(b) Is there a differentiable function g such that $g'(x) = f(x)$ for all $x \in [0, 1]$? (Explain your answer.)

Solution:

Yes. Let $g(x) = \displaystyle\int_0^x f$.

The First Fundamental Theorem of Calculus implies that $g'(x) = f(x)$ for all $x \in [0, 1]$.

(c) Prove from the definition of the integral that $0 \leq \displaystyle\int_0^1 f \leq \frac{5}{8}$.

<u>*Hint*</u>*:* Choose an appropriate partition of $[0, 1]$.

Solution:

Let $P = \{0, \frac{1}{2}, 1\}$. This partition divides the interval $[0,1]$ into two subintervals,

$$I_1 = \big[0, \frac{1}{2}\big] \qquad \text{and} \qquad I_2 = \big[\frac{1}{2}, 1\big] \,.$$

The upper sum of f for P is

$$U(f,P) = M_1\big(\frac{1}{2} - 0\big) + M_2\big(1 - \frac{1}{2}\big) = \frac{M_1 + M_2}{2} \,,$$

where

$$\begin{aligned} M_1 &= \sup\{f(x) : x \in I_1\} \,, \\ \text{and} \quad M_2 &= \sup\{f(x) : x \in I_2\} \,. \end{aligned}$$

Now, since $\sin x \leq 1$ for all $x \in \mathbb{R}$, we have $\sin\frac{1}{x} \leq 1$ for all $x \neq 0$, and therefore

$$f(x) \leq x^2 \,, \qquad \text{for all } x \in \mathbb{R} \,.$$

Consequently, since x^2 is increasing on $[0,1]$, we have

$$\begin{aligned} M_1 &\leq \big(\frac{1}{2}\big)^2 = \frac{1}{4} \\ \text{and} \quad M_2 &\leq \big(1\big)^2 = 1 \end{aligned}$$

and hence

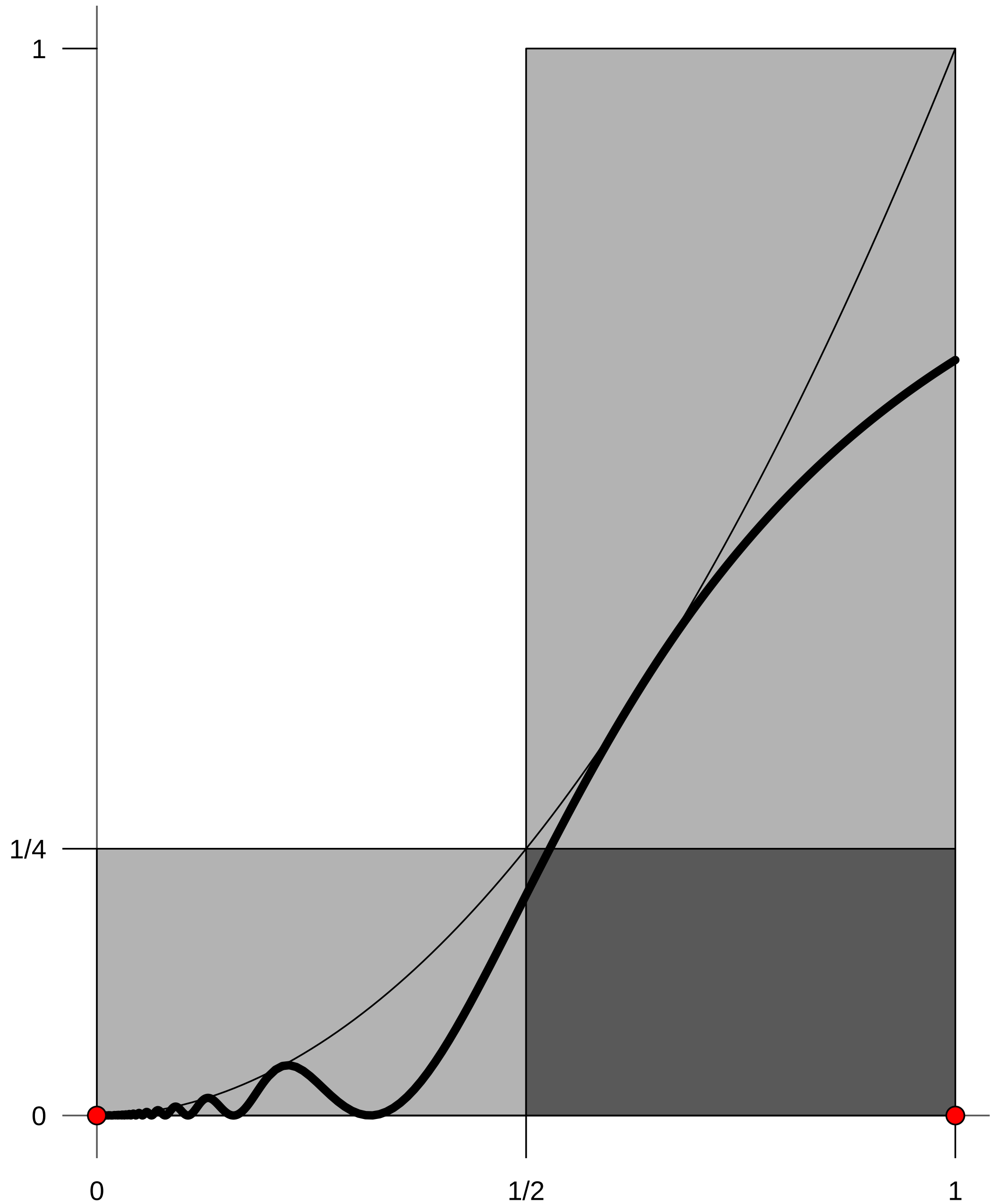
$$U(f,P) \leq \frac{(1/4) + 1}{2} = \frac{5}{8} \,.$$

A simpler analysis shows $L(f,P) \geq 0$, so since $L(f,P) \leq \int_0^1 f \leq U(f,P)$ for <u>any</u> partition P, we have

$$0 \leq \int_0^1 f \leq \frac{5}{8} \,.$$

□

See the next page for a graph that illustrates the argument in the above proof.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1/4
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0
1/2
1