Student Name: Harry Potter Student Number: 33550336

Mathematics 3A03 — Real Analysis I — Prof. David Earn

TERM TEST #1 — 24 October 2016

SOLUTIONS

Duration: 50 minutes

Notes:

- No calculators, notes, scrap paper, or aids of any kind are permitted.
- This test consists of 6 **pages** and includes **8 questions**. Bring any discrepancy to the attention of your instructor or invigilator.
- All questions are to be answered on this test paper. Use the backs of pages if you need more space. The final page is blank to provide extra space for the final question.
- The first 4 questions are *multiple choice*. *Do not justify your answers to these questions.*
- Always write clearly. An answer that cannot be deciphered cannot be marked.
- The marking scheme is indicated in the margin. The maximum total mark is 50.

Question	Mark
1	1
2	3
3	3
4	3
Subtotal	10

Question	Mark
5	9
6	9
7	12
8	10
Subtotal	40

Total	50

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GOOD LUCK and ENJOY!

MARKS

[1] **QUESTION 1.** *In order to obtain any credit for this question, both parts must be answered in clear handwriting at the top of every page of this test.*

(a) What is your name?

(b) What is your student number?

[3] **QUESTION 2.** (*Circle each correct answer.*) Which of the following is a property of the set of real numbers $\mathbb{R}$ but is not a property of the set of rational numbers $\mathbb{Q}$?

(a) there is no largest element;

(b) every non-empty subset has a least element;

(c) $\boxed{\checkmark}$ every non-empty subset that is bounded below has a greatest lower bound;

(d) the set is countable;

(e) none of the above.

[3] **QUESTION 3.** (*Circle each correct answer.*) Which of the following sets is ***countable***?

(a) $\boxed{\checkmark}$ $\varnothing$

(b) $\boxed{\checkmark}$ $\mathbb{Q}$

(c) $\mathbb{R}$

(d) $\mathbb{C}$

(e) $\boxed{\checkmark}$ $\{m\sqrt{2}+n\sqrt{3} : m,n \in \mathbb{N}\}$

[3] **QUESTION 4.** (*Circle each correct answer.*) A set $E \subseteq \mathbb{R}$ is ***compact*** if and only if

(a) $\boxed{\checkmark}$ E is bounded and contains all its accumulation points;

(b) every sequence of points chosen from E has a subsequence that converges;

(c) every open cover of E can be reduced to a countable subcover;

(d) E is dense and countable;

(e) E is both open and closed.

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[9] **QUESTION 5.** For each of the sets E in the table below, answer YES or NO in each column to indicate whether or not E is closed, bounded or dense in $\mathbb{R}$. *Do not justify your answers.*

Set E	**Closed?**	**Bounded?**	**Dense in** $\mathbb{R}$**?**
$[1,3]$	**YES**	**YES**	**NO**
$\mathbb{Q}$	**NO**	**NO**	**YES**
$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$	**NO**	**YES**	**NO**

[9] **QUESTION 6.** For each of the sets E in the table below, fill in the associated point or set in each column, *i.e.*, for each set E state the least upper bound ($\sup(E)$), the interior (E°), and the boundary (∂E). If the requested point or set does not exist, then indicate this with the symbol $\nexists$. *Do not justify your answers.*

E	$\sup(E)$	E°	∂E
$(-1,0) \cup (0,1)$	1	E	$\{-1,0,1\}$
$\mathbb{Q}$	$\nexists$	$\varnothing$	$\mathbb{R}$
$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$	1	$\varnothing$	$E \cup \{0\}$

[12] **QUESTION 7.** Let $\{s_n\}$ be a sequence and L a real number.

(a) State the formal definition of "the sequence $\{s_n\}$ approaches the limit L as $n \to \infty$".

Solution:

Given any $\varepsilon > 0$, there exists a natural number N such that for all $n \geq N$, $|s_n - L| < \varepsilon$.

Shorthand version: $\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\ni\; \forall n \geq N \;\; |s_n - L| < \varepsilon$.

(b) Given that $\lim_{n\to\infty} s_n = L$, use the formal definition to prove that

$$\lim_{n\to\infty} \left(s_n + \frac{1}{n}\right) = L.$$

Solution:

We need to show that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\left(s_n + \frac{1}{n}\right) - L\right| < \varepsilon\,.$$

Let $\varepsilon > 0$ be given. Since $s_n \to L$ as $n \to \infty$, we can choose $N_1 \in \mathbb{N}$ such that

$$|s_n - L| < \frac{\varepsilon}{2}\,, \qquad \text{for all } n \geq N_1\,.$$

In addition, let

$$N_2 = \left\lceil \frac{2}{\varepsilon} \right\rceil + 1\,,$$

which implies that

$$\frac{1}{N_2} \leq \frac{1}{(2/\varepsilon) + 1} < \frac{1}{2/\varepsilon} = \frac{\varepsilon}{2}\,,$$

and hence that

$$\left|\frac{1}{n}\right| = \frac{1}{n} < \frac{\varepsilon}{2}\,, \qquad \text{for all } n \geq N_2\,.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$\begin{aligned}
\left|\left(s_n + \frac{1}{n}\right) - L\right| &= \left|(s_n - L) + \frac{1}{n}\right| \\
&\leq |s_n - L| + \left|\frac{1}{n}\right| \qquad \text{(triangle inequality)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,
\end{aligned}$$

as required. $\square$

[10] **QUESTION 8.** Recall that a sequence is *monotonic* if it is either non-decreasing or non-increasing.

(a) (*Fill in the blanks.*) The *monotone convergence theorem* (MCT) states that a monotonic sequence $\{s_n\}$ is convergent if and only if it is bounded.

(b) Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers with $a_n < b_n$ for all $n \in \mathbb{N}$, so the interval $[a_n, b_n]$ has length $b_n - a_n > 0$ for each n. Suppose the sequence of closed intervals $\big\{[a_n, b_n] : n \in \mathbb{N}\big\}$ is arranged so that (i) each interval is a subinterval of the one preceding it and (ii) the lengths of the intervals shrink to zero, *i.e.*,

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots \qquad \text{and} \qquad \lim_{n\to\infty} \big(b_n - a_n\big) = 0\,.$$

Prove that there is exactly one point that belongs to every interval of the sequence. *Hint:* $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ means that $a_n \le a_{n+1} \le b_{n+1} \le b_n$.

Solution: As pointed out in the hint, $a_n \le a_{n+1}\ \forall n$ so $\{a_n\}$ is non-decreasing, and $b_{n+1} \le b_n\ \forall n$ so $\{b_n\}$ is non-increasing. Moreover, both sequences are bounded since they both lie completely in the first closed interval in the sequence, $[a_1, b_1]$. Therefore, by the MCT both sequences converge, say $a_n \to a$ and $b_n \to b$. In addition, since the lengths of the nested intervals tend to zero, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \big(b_n - a_n\big) \\ &= \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \qquad (\because \text{both sequences converge}) \\ &= b - a\,, \end{aligned}$$

i.e., $a = b$. Thus, both sequences approach the same limit. Let's call it L.

We now need to prove that L is the unique point in all the nested intervals, *i.e.*, $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{L\}$. There are two things we need to show: (i) $L \in [a_n, b_n]\ \forall n$, and (ii) if $x \in [a_n, b_n]\ \forall n$ then $x = L$.

For (i), we must show that $a_n \le L \le b_n\ \forall n$. Suppose $\exists n^* \in \mathbb{N}$ such that $L < a_{n^*}$. Then, since $a_n \le a_{n+1}\ \forall n$, it follows that $L < a_n\ \forall n \ge n^*$. Let $\varepsilon^* = a_{n^*} - L$ and note that $\varepsilon^* > 0$. For all $n \ge n^*$ we have $|a_n - L| \ge |a_{n^*} - L| = \varepsilon^* > 0$. This contradicts $a_n \to L$ because, for example, $\nexists N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon^*/2\ \forall n \ge N$. Therefore $a_n \le L\ \forall n$. Similarly, $L \le b_n\ \forall n$. Thus, $a_n \le L \le b_n\ \forall n$, *i.e.*, $L \in [a_n, b_n]\ \forall n$.

For (ii), suppose $x \in [a_n, b_n]\ \forall n$, and recall that $b_n - a_n \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \ge N$, $b_n - a_n < \varepsilon$. Noting that $b_n - a_n = b_n - x + x - a_n = |b_n - x| + |a_n - x|$ (because $a_n \le x \le b_n$), it follows that $|b_n - x| + |a_n - x| < \varepsilon$. Since $|b_n - x|$ and $|a_n - x|$ are both non-negative, we have, in particular, $|a_n - x| < \varepsilon\ \forall n \ge N$. Therefore, $x = \lim_{n\to\infty} a_n$, so $x = L$ (since limits of sequences are unique if they exist).

$\square$