

26 Integration

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Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

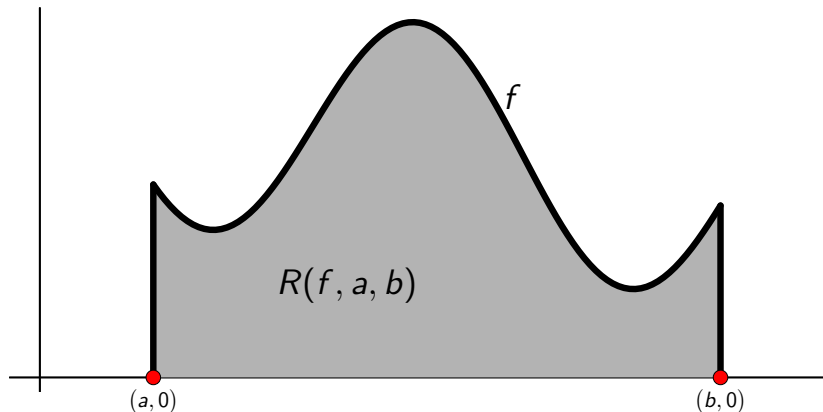
Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 26
Integration
Friday 8 November 2019

Integration

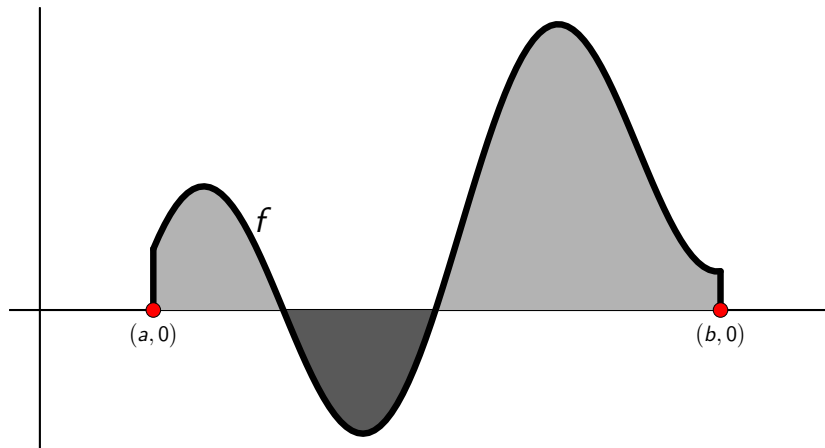
Integration



- “Area of region $R(f, a, b)$ ” is actually a very subtle concept.
- We will only scratch the surface of it.
- Textbook presentation of integral is different (but equivalent).

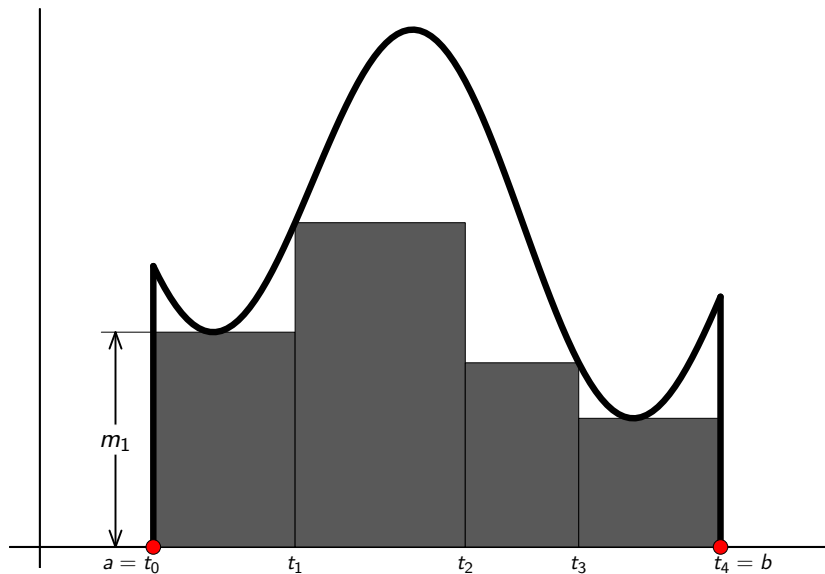
Our treatment is closer to that in M. Spivak “Calculus” (2008).

Integration

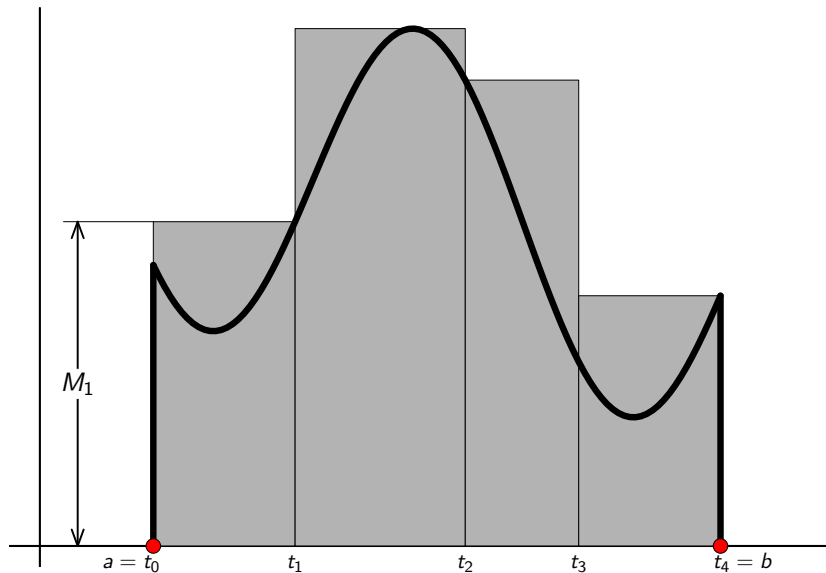


- Contribution to “area of $R(f, a, b)$ ” is positive or negative depending on whether f is positive or negative.

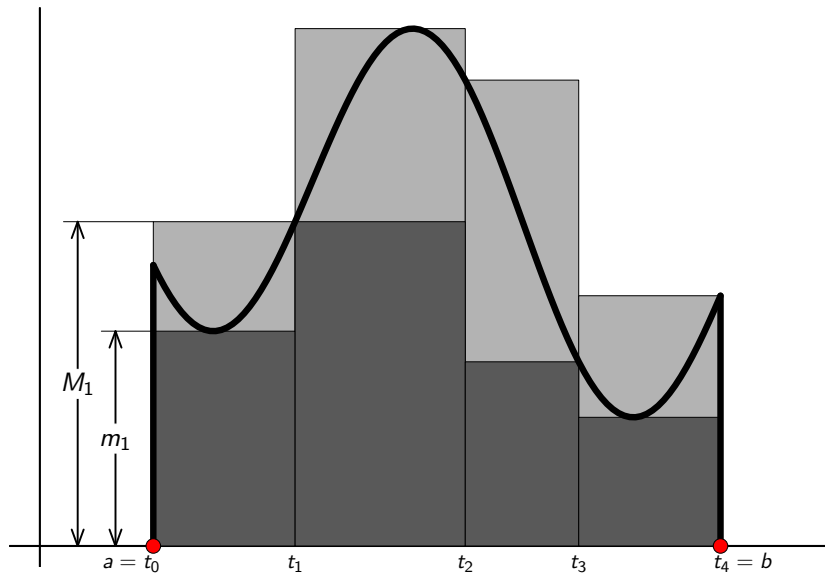
Lower sum



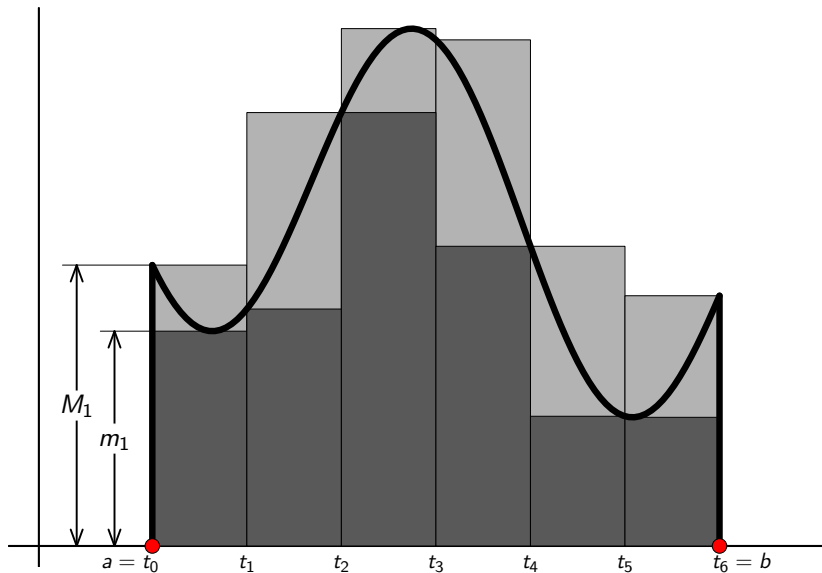
Upper sum



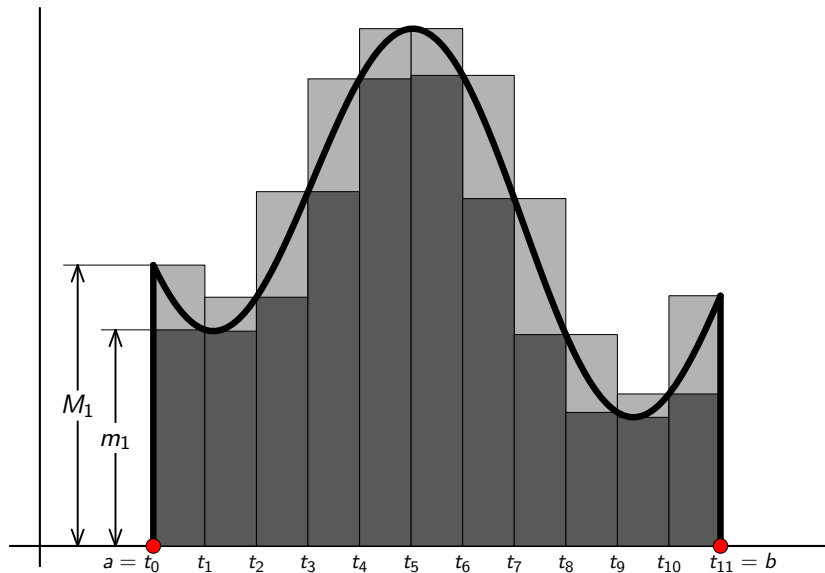
Lower and upper sums



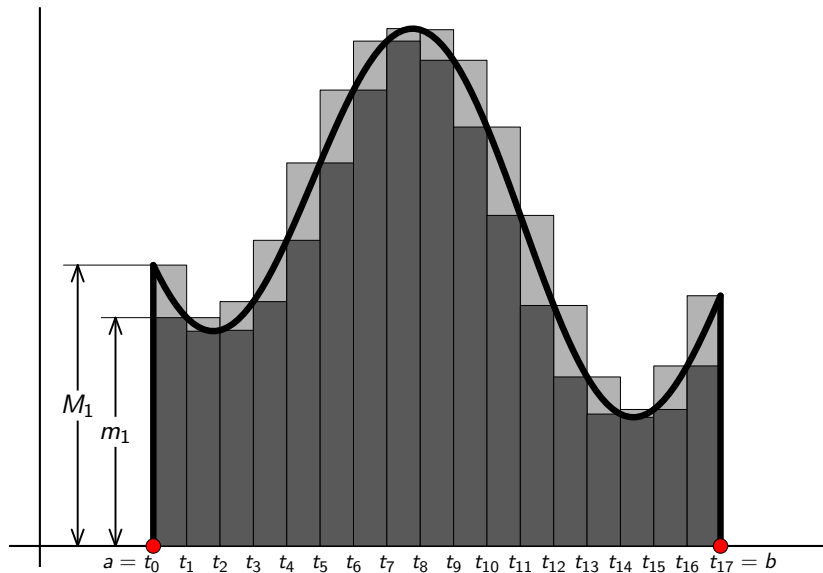
Lower and upper sums



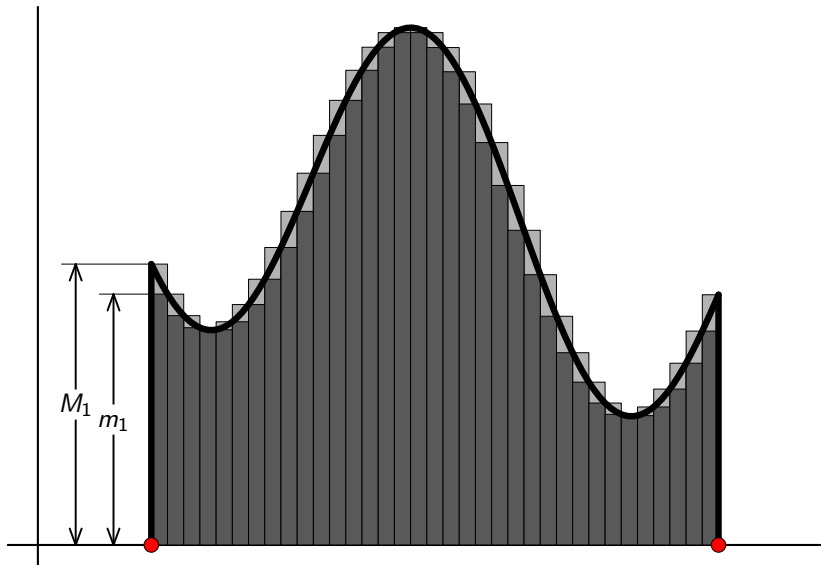
Lower and upper sums



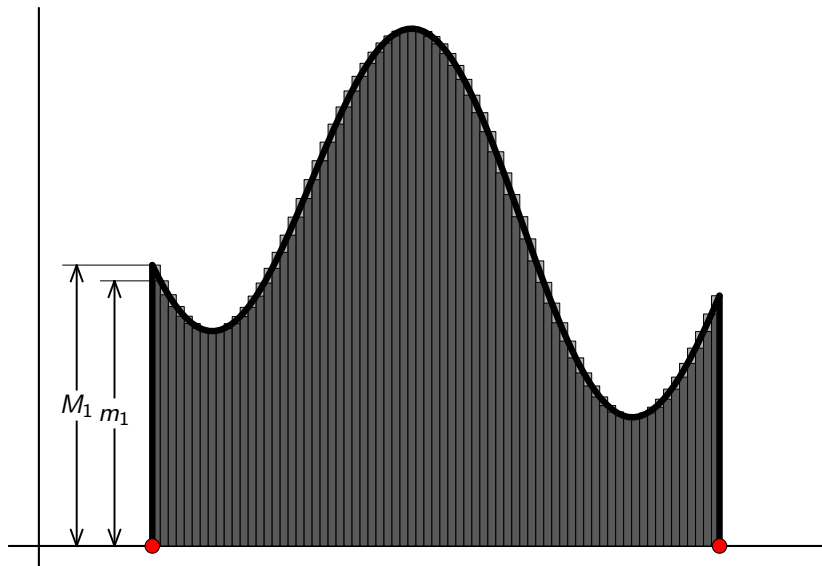
Lower and upper sums



Lower and upper sums



Lower and upper sums



Rigorous development of the integral

Definition (Partition)

Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a , and one of which is b .

We normally label the points in a partition

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

so the i th subinterval in the partition is

$$[t_{i-1}, t_i].$$

Rigorous development of the integral

Definition (Lower and upper sums)

Suppose f is bounded on $[a, b]$ and $P = \{t_0, \dots, t_n\}$ is a **partition** of $[a, b]$. Let

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \},$$

$$M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The **lower sum** of f for P , denoted by $L(f, P)$, is defined as

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}).$$

The **upper sum** of f for P , denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- The **lower and upper sums** correspond to the total areas of rectangles lying below and above the graph of f in our **motivating sketch**.
- However, these sums have been defined precisely without any appeal to a concept of “area”.
- The requirement that f be bounded on $[a, b]$ is essential in order that all the m_i and M_i be well-defined.
- It is also essential that the m_i and M_i be defined as inf's and sup's (rather than maxima and minima) because f was not assumed continuous.

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- Since $m_i \leq M_i$ for each i , we have

$$m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}). \quad i = 1, \dots, n.$$

∴ For any partition P of $[a, b]$ we have

$$L(f, P) \leq U(f, P),$$

because

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}),$$
$$U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1}).$$

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on [Math 3A03](#)
- Click on [Take Class Poll](#)
- Fill in poll **Lecture 26: Lower and Upper Sums**
- .

Rigorous development of the integral

Relationship between motivating sketch and rigorous definition of lower and upper sums:

- More generally, if P_1 and P_2 are any two partitions of $[a, b]$, it ought to be true that

$$L(f, P_1) \leq U(f, P_2),$$

because $L(f, P_1)$ should be \leq area of $R(f, a, b)$, and $U(f, P_2)$ should be \geq area of $R(f, a, b)$.

- But “ought to” and “should be” prove nothing, especially since we haven’t yet even defined “area of $R(f, a, b)$ ”.
- Before we can *define* “area of $R(f, a, b)$ ”, we need to prove that $L(f, P_1) \leq U(f, P_2)$ for any partitions $P_1, P_2 \dots$



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 27
Integration II
Tuesday 12 November 2019

Poll

- Go to https://www.childsmath.ca/childsforms/main_login.php
- Click on **Math 3A03**
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- .

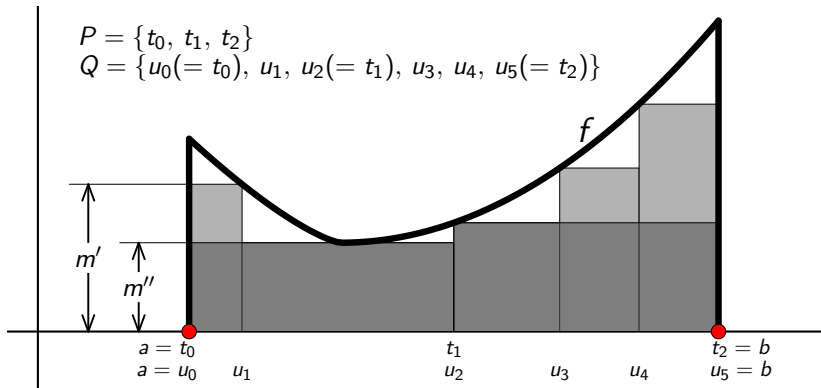
Announcements

- **Assignment 4** was due before class today.
- **Assignment 5** is due on **Thursday 21 November 2019 @ 2:25pm** via **crowdmark**.
- **Math 3A03 Test #2**
Tuesday 26 November 2019, 5:30–7:00pm, in JHE 264
- **Assignment 6** will be due on **Tuesday 3 December 2019 @ 2:25pm** via **crowdmark**.
- **Math 3A03 Final Exam:** Fri 6 Dec 2019, 9:00am–11:30am
Location: MDCL 1105

Rigorous development of the integral

Lemma

If *partition* $P \subseteq$ *partition* Q (i.e., if every point of P is also in Q), then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.



Rigorous development of the integral

Proof of Lemma

As a first step, consider the special case in which the finer partition Q contains only one more point than P :

$$P = \{t_0, \dots, t_n\},$$

$$Q = \{t_0, \dots, t_{k-1}, u, t_k, \dots, t_n\},$$

where

$$a = t_0 < t_1 < \dots < t_{k-1} < u < t_k < \dots < t_n = b.$$

Let

$$m' = \inf \{ f(x) : x \in [t_{k-1}, u] \},$$

$$m'' = \inf \{ f(x) : x \in [u, t_k] \}.$$

... continued ...

Rigorous development of the integral

Proof of Lemma (cont.)

Then
$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1}),$$

and
$$L(f, Q) = \sum_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) \\ + m''(t_k - u) + \sum_{i=k+1}^n m_i(t_i - t_{i-1}).$$

\therefore To prove $L(f, P) \leq L(f, Q)$, it is enough to show

$$m_k(t_k - t_{k-1}) \leq m'(u - t_{k-1}) + m''(t_k - u).$$

... continued ...

Rigorous development of the integral

Proof of Lemma (cont.)

Now note that since

$$\{ f(x) : x \in [t_{k-1}, u] \} \subseteq \{ f(x) : x \in [t_{k-1}, t_k] \},$$

the RHS might contain some additional *smaller* numbers, so we must have

$$\begin{aligned} m_k &= \inf \{ f(x) : x \in [t_{k-1}, t_k] \} \\ &\leq \inf \{ f(x) : x \in [t_{k-1}, u] \} = m'. \end{aligned}$$

Thus, $m_k \leq m'$, and, similarly, $m_k \leq m''$.

$$\begin{aligned} \therefore m_k(t_k - t_{k-1}) &= m_k(t_k - u + u - t_{k-1}) \\ &= m_k(u - t_{k-1}) + m_k(t_k - u) \\ &\leq m'(u - t_{k-1}) + m''(t_k - u), \end{aligned}$$

... continued ...

Rigorous development of the integral

Proof of Lemma (cont.)

which proves (in this special case where Q contains only one more point than P) that $L(f, P) \leq L(f, Q)$.

We can now prove the general case by adding one point at a time.

If Q contains ℓ more points than P , define a sequence of partitions

$$P = P_0 \subset P_1 \subset \cdots \subset P_\ell = Q$$

such that P_{j+1} contains exactly one more point than P_j . Then

$$L(f, P) = L(f, P_0) \leq L(f, P_1) \leq \cdots \leq L(f, P_\ell) = L(f, Q),$$

so $L(f, P) \leq L(f, Q)$.

(Proving $U(f, P) \geq U(f, Q)$ is similar: check!)



Rigorous development of the integral

Theorem (Partition Theorem)

Let P_1 and P_2 be any two partitions of $[a, b]$. If f is bounded on $[a, b]$ then

$$L(f, P_1) \leq U(f, P_2).$$

Proof.

This is a straightforward consequence of the [partition lemma](#).

Let $P = P_1 \cup P_2$, i.e., the partition obtained by combining all the points of P_1 and P_2 .

Then

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$



Rigorous development of the integral

Important inferences that follow from the **partition theorem**:

- For any partition P' , the upper sum $U(f, P')$ is an upper bound for the set of all lower sums $L(f, P)$.

$$\therefore \sup \{L(f, P) : P \text{ a partition of } [a, b]\} \leq U(f, P') \quad \forall P'$$

$$\therefore \sup \{L(f, P)\} \leq \inf \{U(f, P)\}$$

\therefore For any partition P' ,

$$L(f, P') \leq \sup \{L(f, P)\} \leq \inf \{U(f, P)\} \leq U(f, P')$$

- If $\sup \{L(f, P)\} = \inf \{U(f, P)\}$ then we can define “area of $R(f, a, b)$ ” to be this number.

- Is it possible that $\sup \{L(f, P)\} < \inf \{U(f, P)\}$?

Rigorous development of the integral

Example

$\exists? f : [a, b] \rightarrow \mathbb{R}$ such that $\sup \{L(f, P)\} < \inf \{U(f, P)\}$

Let

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b], \\ 0 & x \in \mathbb{Q}^c \cap [a, b]. \end{cases}$$

If $P = \{t_0, \dots, t_n\}$ then $m_i = 0$ ($\because [t_{i-1}, t_i] \cap \mathbb{Q}^c \neq \emptyset$),
and $M_i = 1$ ($\because [t_{i-1}, t_i] \cap \mathbb{Q} \neq \emptyset$).

$\therefore L(f, P) = 0$ and $U(f, P) = b - a$ for any partition P .

$\therefore \sup \{L(f, P)\} = 0 < b - a = \inf \{U(f, P)\}$. □

Can we define “area of $R(f, a, b)$ ” for such a weird function?

Yes, but not in this course!

Rigorous development of the integral

Definition (Integrable)

A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *integrable* on $[a, b]$ if it is bounded on $[a, b]$ and

$$\begin{aligned} & \sup\{L(f, P) : P \text{ a partition of } [a, b]\} \\ & = \inf\{U(f, P) : P \text{ a partition of } [a, b]\}. \end{aligned}$$

In this case, this common number is called the *integral* of f on $[a, b]$ and is denoted

$$\int_a^b f$$

Note: If f is integrable then for any partition P we have

$$L(f, P) \leq \int_a^b f \leq U(f, P),$$

and $\int_a^b f$ is the unique number with this property.

Rigorous development of the integral

- *Notation:*

$$\int_a^b f(x) dx \quad \text{means precisely the same as} \quad \int_a^b f.$$

- The symbol “ dx ” has no meaning in isolation just as “ $x \rightarrow$ ” has no meaning except in $\lim_{x \rightarrow a} f(x)$.
- It is not clear from the definition which functions are **integrable**.
- The definition of the **integral** does not itself indicate how to compute the integral of any given **integrable** function. So far, without a lot more effort we can't say much more than these two things:
 - 1 If $f(x) \equiv c$ then f is **integrable** on $[a, b]$ and $\int_a^b f = c \cdot (b - a)$.
 - 2 The **weird example** function is not **integrable**.

Rigorous development of the integral

- A function that is **integrable** according to our definition is usually said to be **Riemann integrable**, to distinguish this definition from other definitions of integrability.
- In Math 4A03 you will define “Lebesgue integrable”, a more subtle concept that makes it possible to attach meaning to “area of $R(f, a, b)$ ” for the **weird example** function (among others), and to precisely characterize functions that are Riemann integrable.

Rigorous development of the integral

Theorem (Equivalent condition for integrability)

A *bounded* function $f : [a, b] \rightarrow \mathbb{R}$ is *integrable* on $[a, b]$ iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof.

2016 Assignment 5. □

Note: This theorem is just a restatement of the definition of integrability. It is often more convenient to work with $\varepsilon > 0$ than with sup's and inf's.



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 28
Integration III
Thursday 14 November 2019

Last time...

Rigorous development of integral:

- Definition: **integrable**.
- Example: **non-integrable function**.
- Theorem: Equivalent " ϵ - P " definition of integrable.

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on **Math 3A03**
- Click on **Take Class Poll**
- Fill in poll **Lecture 28: Integrable vs Continuous vs Differentiable**
- .

Integral theorems

Theorem

If f is continuous on $[a, b]$ then f is *integrable* on $[a, b]$.

Rough work to prepare for proof:

$$U(f, P) - L(f, P) = \sum_{i=1}^n (M_i - m_i)(t_i - t_{i-1})$$

Given $\varepsilon > 0$, choose a partition P that is so fine that $M_i - m_i < \varepsilon$ for all i . Then

$$U(f, P) - L(f, P) < \varepsilon \sum_{i=1}^n (t_i - t_{i-1}) = \varepsilon(b - a).$$

Not quite what we want. So choose the partition P such that $M_i - m_i < \varepsilon/(b - a)$ for all i . To get that, choose P such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b - a)} \quad \text{if } |x - y| < \max_{1 \leq i \leq n} (t_i - t_{i-1}),$$

which we can do because f is uniformly continuous on $[a, b]$.

Integral theorems

Proof that continuous \implies integrable

Since f is continuous on the compact set $[a, b]$, it is bounded on $[a, b]$ (which is the first requirement to be **integrable** on $[a, b]$).

Also, since f is continuous on the compact set $[a, b]$, it is uniformly continuous on $[a, b]$. $\therefore \forall \varepsilon > 0 \exists \delta > 0$ such that $\forall x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2(b - a)}.$$

Now choose a partition of $[a, b]$ such that the length of each subinterval $[t_{i-1}, t_i]$ is less than δ , i.e., $t_i - t_{i-1} < \delta$. Then, for any $x, y \in [t_{i-1}, t_i]$ we have $|x - y| < \delta$ and therefore

... continued ...

Integral theorems

Proof that continuous \implies integrable (cont.)

$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)} \quad \forall x, y \in [t_{i-1}, t_i].$$

$$\therefore M_i - m_i \leq \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a} \quad i = 1, \dots, n.$$

Since this is true for all i , it follows that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i - m_i)(t_i - t_{i-1}) \\ &< \frac{\varepsilon}{b-a} \sum_{i=1}^n (t_i - t_{i-1}) = \frac{\varepsilon}{b-a} (b-a) = \varepsilon. \end{aligned}$$



Properties of the integral

Theorem (Integral segmentation)

Let $a < c < b$. If f is *integrable* on $[a, b]$, then f is *integrable* on $[a, c]$ and on $[c, b]$. Conversely, if f is *integrable* on $[a, c]$ and $[c, b]$ then f is *integrable* on $[a, b]$. Finally, if f is *integrable* on $[a, b]$ then

$$\int_a^b f = \int_a^c f + \int_c^b f. \quad (\heartsuit)$$

(a good exercise)

This theorem motivates these definitions:

$$\int_a^a f = 0 \quad \text{and} \quad \int_a^b f = -\int_b^a f \quad \text{if } a > b.$$

Then (\heartsuit) holds for any $a, b, c \in \mathbb{R}$.

Properties of the integral

Theorem (Algebra of integrals – a.k.a. \int_a^b is a linear operator)

If f and g are *integrable* on $[a, b]$ and $c \in \mathbb{R}$ then $f + g$ and cf are *integrable* on $[a, b]$ and

$$\mathbf{1} \quad \int_a^b (f + g) = \int_a^b f + \int_a^b g;$$

$$\mathbf{2} \quad \int_a^b cf = c \int_a^b f.$$

(proofs are relatively easy; good exercises)

Theorem (Integral of a product)

If f and g are *integrable* on $[a, b]$ then fg is *integrable* on $[a, b]$.

(proof is much harder; tough exercise)

Properties of the integral

Lemma (Integral bounds)

Suppose f is integrable on $[a, b]$. If $m \leq f(x) \leq M$ for all $x \in [a, b]$ then

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Proof.

For any partition P , we must have $m \leq m_i \forall i$ and $M \geq M_i \forall i$.

$$\therefore m(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a) \quad \forall P$$

$$\therefore m(b - a) \leq \sup\{L(f, P)\} = \int_a^b f = \inf\{U(f, P)\} \leq M(b - a).$$



Properties of the integral

Theorem (Integrals are continuous)

If f is *integrable* on $[a, b]$ and F is defined on $[a, b]$ by

$$F(x) = \int_a^x f,$$

then F is continuous on $[a, b]$.

Proof

Let's first consider $x_0 \in [a, b)$ and show F is continuous from above at x_0 , i.e., $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$. If $x \in (x_0, b]$ then

$$(\heartsuit) \quad \implies \quad F(x) - F(x_0) = \int_a^x f - \int_a^{x_0} f = \int_{x_0}^x f. \quad (*)$$

... continued ...

Properties of the integral

Proof (cont.)

Since f is **integrable** on $[a, b]$, it is bounded on $[a, b]$, so $\exists M > 0$ such that

$$-M \leq f(x) \leq M \quad \forall x \in [a, b],$$

from which the **integral bounds lemma** implies

$$-M(x - x_0) \leq \int_{x_0}^x f \leq M(x - x_0),$$

$$\therefore (*) \implies -M(x - x_0) \leq F(x) - F(x_0) \leq M(x - x_0).$$

\therefore For any $\varepsilon > 0$ we can ensure $|F(x) - F(x_0)| < \varepsilon$ by requiring $0 \leq x - x_0 < \varepsilon/M$, which proves $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$.

A similar argument starting from $x_0 \in (a, b]$ and $x \in [a, x_0)$ yields $\lim_{x \rightarrow x_0^-} F(x) = F(x_0)$. Thus, “integrals are continuous”. \square



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 29
Integration IV
Friday 15 November 2019

Please consider. . .

5 minute *Student Respiratory Illness Survey:*

<https://surveys.mcmaster.ca/limesurvey/index.php/893454>

Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.

–Dr. Marek Smieja (Infectious Diseases)

Announcements

- **Assignment 5** is due on **Thursday 21 November 2019 @ 2:25pm** via **crowdmark**.
- **Math 3A03 Test #2**
Tuesday 26 November 2019, 5:30–7:00pm, in JHE 264
- **Assignment 6** will be due on **Tuesday 3 December 2019 @ 2:25pm** via **crowdmark**.
- **Math 3A03 Final Exam:** Fri 6 Dec 2019, 9:00am–11:30am
Location: MDCL 1105

Last time...

Rigorous development of integral:

- continuous \implies integrable.
- Integral segmentation.
- Algebra of integrals.
- Integral bounds lemma.
- Integrals are continuous.

Fundamental Theorem of Calculus

Theorem (First Fundamental Theorem of Calculus)

Let f be *integrable* on $[a, b]$, and define F on $[a, b]$ by

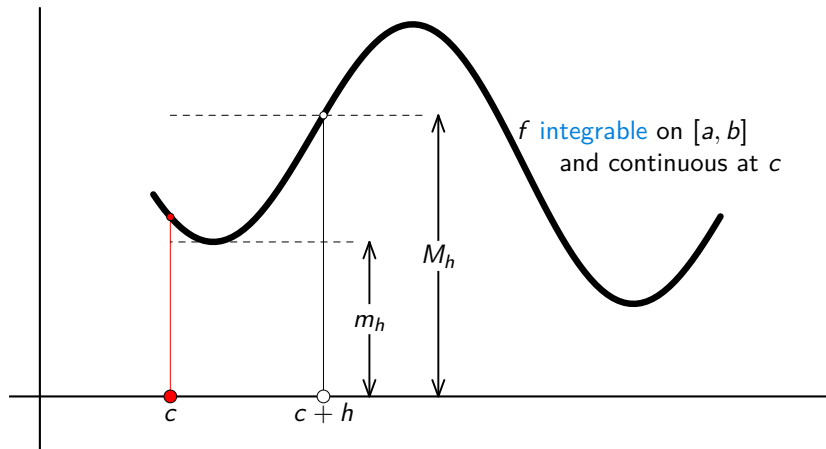
$$F(x) = \int_a^x f.$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c , and

$$F'(c) = f(c).$$

Note: If $c = a$ or b , then $F'(c)$ is understood to mean the right- or left-hand derivative of F .

Fundamental Theorem of Calculus



$$F(c+h) - F(c) \simeq f(c+h) \cdot h$$

and $\lim_{h \rightarrow 0} f(c+h) = f(c)$

$$\implies \lim_{h \rightarrow 0} \frac{F(c+h) - F(c)}{h} = f(c)$$

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus

Suppose $c \in [a, b)$, and $0 < h \leq b - c$. Then the [integral segmentation theorem](#) implies

$$F(c + h) - F(c) = \int_c^{c+h} f.$$

Motivated by the [sketch](#), define

$$m_h = \inf \{ f(x) : x \in [c, c + h] \},$$

$$M_h = \sup \{ f(x) : x \in [c, c + h] \}.$$

Then the [integral bounds lemma](#) implies

$$m_h \cdot h \leq \int_c^{c+h} f \leq M_h \cdot h,$$

... continued ...

Fundamental Theorem of Calculus

Proof of First Fundamental Theorem of Calculus (cont.)

and hence

$$m_h \leq \frac{F(c+h) - F(c)}{h} \leq M_h.$$

This inequality is true for any integrable function. However, because f is continuous at c , we have

$$\lim_{h \rightarrow 0^+} m_h = \lim_{h \rightarrow 0^+} M_h = f(c),$$

so the squeeze theorem implies

$$F'_+(c) = \lim_{h \rightarrow 0^+} \frac{F(c+h) - F(c)}{h} = f(c).$$

A similar argument for $c \in (a, b]$ and $c - a \leq h < 0$ yields $F'_-(c) = f(c)$. □

Fundamental Theorem of Calculus

Corollary

If f is continuous on $[a, b]$ and $f = g'$ for some function g , then

$$\int_a^b f = g(b) - g(a).$$

Proof.

Let $F(x) = \int_a^x f$. Then $\forall x \in [a, b]$, $F'(x) = f(x)$ (by FFTC).
 $\implies F' = f = g'$.

$\therefore \exists c \in \mathbb{R}$ such that $F = g + c$ (2016 Assignment 5).

$\therefore F(a) = g(a) + c$. But $F(a) = \int_a^a f = 0$, so $c = -g(a)$.

$\therefore F(x) = g(x) - g(a)$.

This is true, in particular, for $x = b$, so $\int_a^b f = g(b) - g(a)$. \square

Poll

- Go to https://www.childsmath.ca/childs/forms/main_login.php
- Click on **Math 3A03**
- Click on **Take Class Poll**
- Fill in poll **Lecture 29: Fundamental Theorem of Calculus**
- .

Fundamental Theorem of Calculus

Theorem (Second Fundamental Theorem of Calculus)

If f is *integrable* on $[a, b]$ and $f = g'$ for some function g , then

$$\int_a^b f = g(b) - g(a).$$

Notes:

- This looks like the [corollary](#) to the [first fundamental theorem](#), except that f is only assumed [integrable](#), not continuous.
- Recall from [Darboux's theorem](#) that if $f = g'$ for some g then f has the [intermediate value property](#), but f need not be continuous.
- g' exists on $[a, b] \implies$ Mean Value Theorem applies to g .
- The proof of the [second fundamental theorem](#) is completely different from the [corollary](#) to the first, because we cannot use the [first fundamental theorem](#) (which assumed f is continuous).

Fundamental Theorem of Calculus

Proof of Second Fundamental Theorem of Calculus

Let $P = \{t_0, \dots, t_n\}$ be any partition of $[a, b]$. By the Mean Value Theorem, for each $i = 1, \dots, n$, $\exists x_i \in [t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1}).$$

Define m_i and M_i as usual. Then $m_i \leq f(x_i) \leq M_i \forall i$, so

$$m_i(t_i - t_{i-1}) \leq f(x_i)(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1}),$$

$$\text{i.e., } m_i(t_i - t_{i-1}) \leq g(t_i) - g(t_{i-1}) \leq M_i(t_i - t_{i-1}).$$

$$\therefore \sum_{i=1}^n m_i(t_i - t_{i-1}) \leq \sum_{i=1}^n (g(t_i) - g(t_{i-1})) \leq \sum_{i=1}^n M_i(t_i - t_{i-1})$$

$$\text{i.e., } L(f, P) \leq g(b) - g(a) \leq U(f, P)$$

for any partition P . $\therefore g(b) - g(a) = \int_a^b f.$



What useful things can we do with integrals?

- Compute areas of complicated shapes: find anti-derivatives and use the **second fundamental theorem of calculus**.
- Define trigonometric functions (rigorously).
- Define logarithm and exponential functions (rigorously).



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 35
Integration V
Friday 29 November 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on **Math 3A03**
- Click on **Take Class Poll**
- Fill in poll **Lecture 35: Final Exam time and place**
- .

Please consider. . .

5 minute *Student Respiratory Illness Survey:*

<https://surveys.mcmaster.ca/limesurvey/index.php/893454>

Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.

–Dr. Marek Smieja (Infectious Diseases)

Fall 2019 Course Evaluations

Open: Thursday November 21, 10:00AM

Close: Thursday December 5, 11:59PM

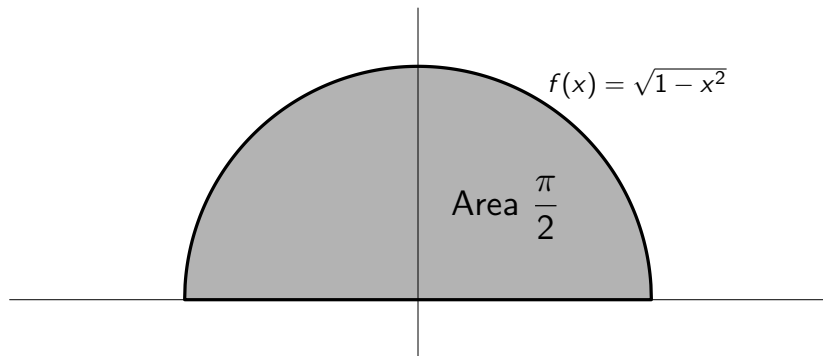
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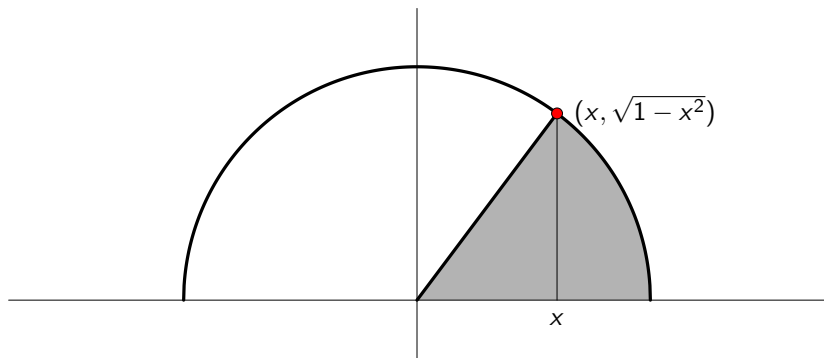
You can monitor what percentage of students has completed the course evaluation at <https://evals.mcmaster.ca/stats/index.php?fac=SCIENCE>.

What is π ?

Definition

$$\pi \equiv 2 \int_{-1}^1 \sqrt{1-x^2} \, dx .$$

What are cos and sin ?

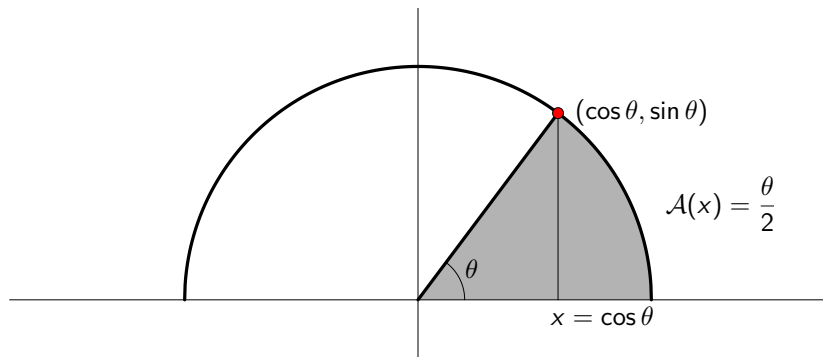


Definition (Sectoral area)

If $x \in [-1, 1]$ then
$$\mathcal{A}(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2} dt.$$

Note: $\mathcal{A}(-1) = \pi/2$, $\mathcal{A}(1) = 0$.

What are cos and sin ?



Length of circular arc swept out by angle θ : θ

Area of sectoral region swept out by angle θ : $\theta/2$

So, if $\theta \in [0, \pi]$ then we define $\cos \theta$ to be the unique number in $[-1, 1]$ such that $\mathcal{A}(\cos \theta) = \theta/2$, and we define $\sin \theta$ to be $\sqrt{1 - (\cos \theta)^2}$.

We must prove: given $x \in [0, \pi] \exists! y \in [-1, 1]$ such that $\mathcal{A}(y) = x/2$.

What are cos and sin ?

Proof that $\forall x \in [0, \pi] \exists! y \in [-1, 1]$ such that $\mathcal{A}(y) = x/2$:

Existence: $\mathcal{A}(1) = 0$, $\mathcal{A}(-1) = \pi/2$, and \mathcal{A} is continuous. Hence by the **intermediate value theorem** $\exists y \in [-1, 1]$ such that $\mathcal{A}(y) = x/2$.

Uniqueness: \mathcal{A} is differentiable on $(-1, 1)$ and $\mathcal{A}'(x) < 0$ on $(-1, 1)$.
 \therefore On $(-1, 1)$, \mathcal{A} is decreasing, and hence one-to-one.

Definition (cos and sin)

If $x \in [0, \pi]$ then $\cos x$ is the unique number in $[-1, 1]$ such that $\mathcal{A}(\cos x) = x/2$, and $\sin x = \sqrt{1 - (\cos x)^2}$.

These definitions are easily extended to all of \mathbb{R} :

- For $x \in [\pi, 2\pi]$, define $\cos x = \cos(2\pi - x)$ and $\sin x = -\sin(2\pi - x)$.
- Then, for $x \in \mathbb{R} \setminus [0, 2\pi]$ define $\cos x = \cos(x \bmod 2\pi)$ and $\sin x = \sin(x \bmod 2\pi)$.

Trigonometric theorems

Given the **rigorous definition of cos and sin**, we can prove:

- 1 **cos** and **sin** are differentiable on \mathbb{R} . Moreover, $\cos' = -\sin$ and $\sin' = \cos$.
- 2 **sec**, **tan**, **csc** and **cot** can all be defined in the usual way and have all the usual properties.
- 3 The **inverse function theorem** allows us to define and compute the derivatives of all the inverse trigonometric functions.
- 4 If f is twice differentiable on \mathbb{R} , $f'' + f = 0$, $f(0) = a$ and $f'(0) = b$, then $f = a \cos + b \sin$.
- 5 For all $x, y \in \mathbb{R}$,

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y, \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y.\end{aligned}$$

Something deep that you know enough to prove

Extra Challenge Problem:
Prove that π is irrational.

What are log and exp ?

Consider the function

$$f(x) = 10^x .$$

What exactly is this function?

In our mathematically naïve previous life, we just assumed that $f(x)$ is well-defined $\forall x \in \mathbb{R}$, and that f has a well-defined inverse function,

$$f^{-1}(x) = \log_{10}(x) .$$

But how are 10^x and $\log_{10}(x)$ defined for irrational x ?

Let's review what we know...

What are log and exp ?

$$n \in \mathbb{N} \implies 10^n = \underbrace{10 \cdots 10}_{n \text{ times}}$$

$$n, m \in \mathbb{N} \implies 10^n \cdot 10^m = 10^{n+m}$$

When we extend 10^x to $x \in \mathbb{Q}$, we want this product rule to be preserved:

$$10^0 \cdot 10^n = 10^{0+n} = 10^n \implies 10^0 = 1$$

$$10^{-n} \cdot 10^n = 10^0 = 1 \implies 10^{-n} = \frac{1}{10^n}$$

$$\underbrace{10^{1/n} \cdots 10^{1/n}}_{n \text{ times}} = 10^{\underbrace{1/n \cdots 1/n}_{n \text{ times}}} = 10^1 = 10 \implies 10^{1/n} = \sqrt[n]{10}$$

What are log and exp ?

Finally, to define 10^q for all $q \in \mathbb{Q}$, note that we must have

$$\left(10^{\frac{1}{n}}\right)^m = \underbrace{10^{\frac{1}{n}} \cdots 10^{\frac{1}{n}}}_{m \text{ times}} = 10^{\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ times}}} = 10^{\frac{m}{n}} \quad \implies \quad 10^{\frac{m}{n}} \stackrel{\text{def}}{=} \left(\sqrt[n]{10}\right)^m$$

Now we're stuck. *How do we extend this scheme to irrational x ?*

We need a more sophisticated idea.

Let's try to find a function on all of \mathbb{R} that satisfies

$$f(x+y) = f(x) \cdot f(y), \quad \forall x, y \in \mathbb{R},$$

and $f(1) = 10.$

It then follows that $f(0) = 1$ and, $\forall x \in \mathbb{Q}$, $f(x) = [f(1)]^x.$

What additional properties can we impose on $f(x)$ that will lead us to a sensible definition of $f(x)$ for all $x \in \mathbb{R}$?

What are log and exp ?

One approach is to insist that f is *differentiable*.

Then we can compute

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h} \\ &= f(x) \cdot \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} = f(x) \cdot f'(0) \equiv \alpha f(x) \end{aligned}$$

So $f'(x) = \alpha f(x)$, *i.e.*, we have f' in terms of unknowns f and α .
So what?!?

Let's look at the inverse function, f^{-1} (think "log₁₀"):

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\alpha f(f^{-1}(x))} = \frac{1}{\alpha x}$$

Holy \$#@%! We have a simple formula for the derivative of f^{-1} !

What are log and exp ?

Since we want $\log_{10} 1 = 0$, we should define $\log_{10} x$ as $(1/\alpha) \int_1^x t^{-1} dt$. *Great idea, but we don't know what α is.*

So, let's ignore α . . .

(and hope that what we end up with is log to some "natural" base).

Definition (Logarithm function)

If $x > 0$ then

$$\log x = \int_1^x \frac{1}{t} dt .$$

This function is strictly increasing ($\log'(x) > 0$ for all $x > 0$) so we can now define:

Definition (Exponential function)

$$\exp = \log^{-1} .$$

What are log and exp ?

With these rigorous definitions of **log** and **exp**, we can prove the following as theorems:

- 1 If $x, y > 0$ then $\log(xy) = \log x + \log y$.
- 2 If $x, y > 0$ then $\log(x/y) = \log x - \log y$.
- 3 If $n \in \mathbb{N}$ and $x > 0$ then $\log(x^n) = n \log x$.
- 4 For all $x \in \mathbb{R}$, $\exp'(x) = \exp(x)$.
- 5 For all $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x) \cdot \exp(y)$.
- 6 For all $x \in \mathbb{Q}$, $\exp(x) = [\exp(1)]^x$.

The last theorem above motivates:

Definition

$$e = \exp(1),$$
$$e^x = \exp(x) \quad \text{for all } x \in \mathbb{R}.$$

What are log and exp ?

We can now give a rigorous definition of 10^x for any $x \in \mathbb{R}$.
In fact, we can do this for any $a > 0$.

Definition (a^x)

If $a > 0$ and x is any real number then

$$a^x = e^{x \log a}.$$

We then have the following theorems for any $a > 0$:

- 1** $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$;
- 2** $a^0 = 1$; $a^1 = a$;
- 3** $a^{x+y} = a^x \cdot a^y$ for all $x, y \in \mathbb{R}$;
- 4** $a^{-x} = 1/a^x$ for all $x \in \mathbb{R}$;
- 5** if $a > 1$ then a^x is increasing on \mathbb{R} ;
- 6** if $0 < a < 1$ then a^x is decreasing on \mathbb{R} .

Using the integral to define useful functions rigorously

- Just as we defined 10^x via the definition of $\log x = \int_1^x \frac{1}{t} dt$, we could have defined the trigonometric functions starting from

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}} dt, \quad -1 < x < 1,$$

rather than the more complicated definition of \cos via $\mathcal{A}(x)$. Many common functions are defined as integrals of rational functions of square roots.

- Any compositions of trig functions, log, exp, rational functions and radicals, are called *elementary functions*.
- Most functions that turn up a lot in applications can be defined rigorously via integrals of elementary functions. Such functions are collectively called *special functions*.



Mathematics
and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03 Real Analysis I

Instructor: David Earn

Lecture 36
Pre-exam Q&A
Tuesday 3 December 2019

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on [Math 3A03](#)
- Click on [Take Class Poll](#)
- Fill in poll **Lecture 36: Final in-class poll!**
- .

Please consider. . .

5 minute *Student Respiratory Illness Survey:*

<https://surveys.mcmaster.ca/limesurvey/index.php/893454>

Please complete this anonymous survey to help us monitor the patterns of respiratory illness, over-the-counter drug use, and social contact within the McMaster community. There are no risks to filling out this survey, and your participation is voluntary. You do not need to answer any questions that make you uncomfortable, and all information provided will be kept strictly confidential. Thanks for participating.

–Dr. Marek Smieja (Infectious Diseases)

Announcements

- **Assignment 6** was due before class today via [crowdmark](#). Solutions will be posted after class.
- Math 3A03 **Final Exam**: Fri 6 Dec 2019, 9:00am–11:30am
Location: MDCL-1105
- Fletcher Gates will be in the Math Help Centre TODAY, 3:30–5:30pm.

Fall 2019 Course Evaluations

Open: Thursday November 21, 10:00AM

Close: Thursday December 5, 11:59PM

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Last time...

- Rigorous definition of trig functions.
- Rigorous definition of log and exp functions.

Today...

- Prove e is irrational.
- Talk about final exam.
- Q&A.

Approximation by Polynomial Functions

Definition (Taylor polynomial)

If f is n times differentiable at a then the **Taylor polynomial of degree n for f at a** is

$$P_{n,a}(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Theorem (Taylor's theorem)

Suppose $f', \dots, f^{(n+1)}$ are defined on $[a, x]$, and that $R_{n,a}(x)$ is defined by $f(x) = P_{n,a}(x) + R_{n,a}(x)$. Then

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x - a)^{n+1}, \quad \text{for some } t \in (a, x).$$

Note: The form of the remainder term here is known as the **Lagrange form** of the remainder.

Approximation by Polynomial Functions

Example (Approximating e)

Use **Taylor's theorem** to show that e can be approximated to within $\frac{3}{(n+1)!}$ for any given n . Also show that $2 < e < 3$.

Recall $e = e^1 = \exp(1)$. $\therefore e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n$, where $R_n = \frac{e^t}{(n+1)!}$ for some $t \in (0, 1)$. Since e^x is increasing on $(0, 1)$, we must have $e^t < e$, so $\frac{1}{(n+1)!} < R_n < \frac{e}{(n+1)!}$. But we can't estimate e using e .

Recall $1 = \log e = \int_1^e \frac{1}{t} dt$, and note $\log 4 = \int_1^4 \frac{1}{t} dt > 1$, since $\frac{1}{2}(2-1) + \frac{1}{4}(4-2) = 1$ is a lower sum for $f(t) = 1/t$ on $[1, 4]$.

$\therefore \log e < \log 4$, i.e., $e < 4$. (*Similarly:* Use $\int_1^2 \frac{1}{t} dt$ to get $e > 2$.)

$\therefore 2 < e < 4$ and $R_n < \frac{4}{(n+1)!}$. Great, but what we actually want is
 $2 < e < 3$ and $R_n < \frac{3}{(n+1)!}$ continued ...

Approximation by Polynomial Functions

Example (Approximating e (cont.))

Given $R_n < \frac{4}{(n+1)!}$, note that for $n = 4$ we have

$$0 < R_n < \frac{4}{5!} = \frac{1}{30},$$

so applying [Taylor's theorem](#) with $n = 4$ we get

$$\begin{aligned} e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + R_n = \left(2 + \frac{17}{24}\right) + R_n \\ &< \left(2 + \frac{17}{24}\right) + \frac{1}{30} < 3. \end{aligned}$$

Thus $e < 3$, and consequently

$$R_n < \frac{e}{(n+1)!} \implies R_n < \frac{3}{(n+1)!}.$$



e is irrational

Theorem (e is irrational)

$\nexists k, m \in \mathbb{N}$ such that $e = k/m$.

Proof.

Suppose $e = k/m$ with $k, m \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we have

$$\frac{k}{m} = e^1 = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n, \quad 0 < R_n < \frac{3}{(n+1)!}.$$

$$\therefore \frac{n!k}{m} = n! + n! + \frac{n!}{2!} + \cdots + \frac{n!}{n!} + n!R_n, \quad n \in \mathbb{N}.$$

This is true, in particular, for $n > 3$ and $n > m$, in which case every term in this equation other than $n!R_n$ is an integer. So $n!R_n$ is also an integer! But $0 < R_n < 3/(n+1)!$, so since $n > 3$ we have

$$0 < n!R_n < \frac{3}{n+1} < \frac{3}{4} < 1,$$

which is impossible for an integer. □

Final Exam this Friday 6 Dec 2019, 9:00am–11:30am,
MDCL-1105

Let's have a look at the Final Exam...

Fall 2019 Course Evaluations

Open: Thursday November 21, 10:00AM

Close: Thursday December 5, 11:59PM

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