34 Wrap-up

Poll

- Go to https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
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Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 34
Wrap-up
Monday 7 April 2025

Announcements

- Poll results on the web site are up-to-date as of this morning before class.
- Solutions to Assignment 5 have been updated. There is now a full proof of the Jordan-von Neumann theorem, which we will discuss today.
- We'll also discuss the final exam today.

Which norms are induced by inner products?

Pascual Jordan and John von Neumann (1935, "On Inner Products in Linear, Metric Spaces", Annals of Mathematics 36(3), 719–723).

Norms induced by inner products

Theorem (Jordan and von Neumann (1935))

A norm $\|\cdot\|$ on a real vector space V is induced by an inner product if and only if it satisfies the ***parallelogram law****,*

$$\|x+y\|^2 + \|x-y\|^2 \;=\; 2\,\|x\|^2 + 2\,\|y\|^2\,, \quad \forall x, y \in V. \qquad \text{(PL)}$$

Note: Jordan & von Neumann proved this for vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

Corollary

The Euclidean norm is the only p-norm on $\mathbb{R}^n$, ℓ^∞, or $C[a,b]$ that is induced by an inner product.

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

$(\implies)$

Given an inner product $\langle \cdot, \cdot \rangle$ on a vector space V, and the induced norm $\|\cdot\|$, for any $x, y \in V$ we have

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \Big(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \Big) \\
&\qquad + \Big(\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \Big) \\
&= 2\langle x,x\rangle + 2\langle y,y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2 ,
\end{aligned}$$

as required.

That was the easy part...

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

$(\impliedby)$

First, note that (changing a sign in the calculation on the previous slide) given an inner product $\langle\cdot,\cdot\rangle$, and the induced norm $\|\cdot\|$, for any $x, y \in V$, we have

$$\begin{aligned}
\|x+y\|^2 - \|x-y\|^2 &= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \\
&= \Big(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \Big) \\
&\qquad - \Big(\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \Big) \\
&= 2\langle x,y\rangle + 2\langle y,x\rangle \\
&= 4\langle x,y\rangle\,.
\end{aligned}$$

Solving for $\langle x, y\rangle$, we see that if a norm is induced by an inner product, then the inner product can be expressed using the norm via

$$\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \qquad (\heartsuit)$$

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

Thus, if there is an inner product $\langle\cdot,\cdot\rangle$ that induces a given norm $\|\cdot\|$, then the inner product must be given by (♡).

We now need to show that, given any norm, if we define $\langle\cdot,\cdot\rangle$ via (♡) then $\langle\cdot,\cdot\rangle$ is actually an inner product if and only if the norm satisfies the parallelogram law (PL). To that end, we examine each of the axioms of an inner product, assuming $\langle\cdot,\cdot\rangle$ is defined from a given norm $\|\cdot\|$ via (♡). *This is not an easy problem. If you solved it, you should be proud!*

So, suppose $\|\cdot\|$ is a norm on V that satisfies the parallelogram law (PL), and that $\langle\cdot,\cdot\rangle$ is a candidate inner product defined by (♡).

Symmetry.
Swapping x and y, in the definition (♡) yields

$$\langle y, x\rangle = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2\right) = \langle x, y\rangle. \tag{1}$$

Thus, the inner product is symmetric.

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

Positive Definiteness

By definition,

$$\langle x, x\rangle = \frac{1}{4}\left(\|x+x\|^2 - \|x-x\|^2\right) = \frac{1}{4}\left(\|2x\|^2 - 0\right) = \frac{1}{4}\left(4\|x\|^2\right) = \|x\|^2.$$

Since norms are always nonnegative and zero if and only if $x = 0$, we conclude $\langle x, x\rangle = 0$ if and only if $x = 0$.

Thus, $\langle x, x\rangle$ is positive definite.

Linearity in the First Argument

This is the very challenging part. We will break it into two pieces: $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle ax, z\rangle = a\,\langle x, z\rangle$. Both are tricky, but—perhaps suprisingly—the second requires more analytical creativity.

To begin with, it is worth making a note of what the paralleogram law (PL) implies if one of the two vectors is the zero vector. If $x = 0$, then (PL) states that $\|y\|^2 + \|-y\|^2 = 2\,\|y\|^2$, and hence[1]

$$\|-y\|^2 = \|y\|^2 \quad \forall y \in V. \tag{2}$$

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

Consequently, if we insert $x = 0$ in ($\heartsuit$), then we we find

$$\langle 0, y\rangle = 0 \quad \forall y \in V. \tag{3}$$

This may seem an obvious fact, but remember that we do not yet know that $\langle\cdot,\cdot\rangle$ is an inner product. All we know is that it is a function of two variables defined by ($\heartsuit$) and that the norm satisfies (PL).

Now, the parallelogram law (PL) is presumed to hold for all $x, y \in V$. So, it is true if we write x and y as any linear combinations of other vectors in V. *Doing this judiciously will get us where we want to go.* Given any $u, v, w \in V$, if we let $x = u + v$ and $y = w$ then (PL) states that

$$\|u+v+w\|^2 + \|u+v-w\|^2 = 2\,\|u+v\|^2 + 2\,\|w\|^2 . \tag{4}$$

Similarly, if we let $x = u - v$ and $y = w$ then (PL) states that

$$\|u-v+w\|^2 + \|u-v-w\|^2 = 2\,\|u-v\|^2 + 2\,\|w\|^2 . \tag{5}$$

If we now subtract Eq. (5) from Eq. (4) we obtain

$$\begin{aligned} &\|u+v+w\|^2 + \|u+v-w\|^2 - \|u-v+w\|^2 - \|u-v-w\|^2 \\ &\quad = 2\,\|u+v\|^2 - 2\,\|u-v\|^2 . \end{aligned} \tag{6}$$

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

Recalling the definition of the candidate inner product (♡), the RHS of Eq. (6) is

$$2\|u+v\|^2 - 2\|u-v\|^2 \;=\; 8\langle u, v\rangle \;,$$

and the LHS of Eq. (6) is

$$\begin{aligned}
&\|(u+v)+w\|^2 + \|(u+v)-w\|^2 - \|(u-v)+w\|^2 - \|(u-v)-w\|^2 \\
&\quad= \;\|(u+w)+v\|^2 + \|(u-w)+v\|^2 - \|(u+w)-v\|^2 - \|(u-w)-v\|^2 \\
&\quad= \;\|(u+w)+v\|^2 - \|(u+w)-v\|^2 + \|(u-w)+v\|^2 - \|(u-w)-v\|^2 \\
&\quad= \;4\langle u+w, v\rangle + 4\langle u-w, v\rangle \;.
\end{aligned}$$

Therefore, Eq. (6) can be written

$$\langle u+w, v\rangle + \langle u-w, v\rangle \;=\; 2\langle u, v\rangle \;. \tag{7}$$

It would help us if we could replace $2\langle u, v\rangle$ with $\langle 2u, v\rangle$ in this equation, but we don't (yet) know that scalar multiples can be brought inside the first argument of $\langle\cdot,\cdot\rangle$. Nevertheless, we can see that this is true if the scalar multiple happens to be 2, by considering the special case of Eq. (7) with $u = w$, which yields

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

$$\langle 2u, v\rangle + \langle 0, v\rangle \;=\; 2\,\langle u, v\rangle\ , \tag{8}$$

and, recalling Eq. (3), this becomes

$$\langle 2u, v\rangle \;=\; 2\,\langle u, v\rangle\ . \tag{9}$$

Thus, in Eq. (7), we can indeed replace $2\,\langle u, v\rangle$ with $\langle 2u, v\rangle$ to obtain

$$\langle u + w, v\rangle + \langle u - w, v\rangle \;=\; \langle 2u, v\rangle\ . \tag{10}$$

This is true for any $u, v, w \in V$, so, given any $x, y, z \in V$ (unrelated to any particular x, y, z we started with), if we insert

$$u = \frac{1}{2}(x + y), \qquad v = z, \qquad w = \frac{1}{2}(x - y), \tag{11}$$

in Eq. (10) we obtain

$$\langle x, z\rangle + \langle y, z\rangle \;=\; \langle x + y, z\rangle\ , \quad \forall x, y, z \in V, \tag{12}$$

as required.

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

Now the really tricky part: we need to prove that for any $x, y \in V$ and any $a \in \mathbb{R}$, $\langle ax, y\rangle = a\langle x, y\rangle$. Our first step is to observe that Eq. (9) is, in fact, a special case of what we are now aiming to prove, namely the case $a = 2$. Using Eq. (9) and Eq (12), it follows that

$$\langle 3u, v\rangle \;=\; \langle 2u + u, v\rangle \;=\; \langle 2u, v\rangle + \langle u, v\rangle \;=\; 2\langle u, v\rangle + \langle u, v\rangle \;=\; 3\langle u, v\rangle\,.$$

Similarly, (formally by induction) we have

$$\langle nu, v\rangle \;=\; n\langle u, v\rangle \qquad \forall n \in \mathbb{N}. \tag{13}$$

Now consider a rational number $\frac{m}{n}$, with $m, n \in \mathbb{N}$. We have

$$\left\langle \tfrac{m}{n}u, v\right\rangle \;=\; m\left\langle \tfrac{1}{n}u, v\right\rangle \tag{14a}$$

$$\implies \quad n\left\langle \tfrac{m}{n}u, v\right\rangle \;=\; nm\left\langle \tfrac{1}{n}u, v\right\rangle \;=\; m\langle u, v\rangle \tag{14b}$$

$$\implies \quad \left\langle \tfrac{m}{n}u, v\right\rangle \;=\; \tfrac{m}{n}\langle u, v\rangle\,, \tag{14c}$$

so $\langle au, v\rangle = a\langle u, v\rangle$ for any positive rational number.

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

If we now observe, using Eqs. (3) and (12), that

$$0 \;=\; \langle 0, v\rangle \;=\; \langle u + (-u), v\rangle \;=\; \langle u, v\rangle + \langle -u, v\rangle\ , \tag{15}$$

then we have

$$\langle -u, v\rangle \;=\; -\langle u, v\rangle \qquad \forall u, v \in V, \tag{16}$$

which then implies that, in fact, $\langle au, v\rangle = a\langle u, v\rangle$ for all $a \in \mathbb{Q}$.

Another way of saying this is that, for any $u, v \in V$, the function

$$f(a) \;=\; \langle au, v\rangle - a\langle u, v\rangle \tag{17}$$

satisfies $f(a) = 0$ for all $a \in \mathbb{Q}$. If we can now show that f is a continuous function of a for all $a \in \mathbb{R}$, then it will follow that $f(a) = 0$ for all $a \in \mathbb{R}$. To that end, first recall that in any normed vector space V,

$$\big|\|x\| - \|y\|\big| \;\le\; \|x - y\| \qquad \forall x, y \in V \tag{18}$$

(which you can prove by noting that $\|x\| = \|(x - y) + y\|$ and using the triangle inequality). Consequently, if $\langle \cdot, \cdot\rangle$ is defined by ($\heartsuit$) then

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

$$
\begin{aligned}
|\langle x, y\rangle| &= \tfrac{1}{4}\Big|\,\|x+y\|^2 - \|x-y\|^2\,\Big| \\
&= \tfrac{1}{4}\Big|\big(\,\|x+y\| - \|x-y\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| \\
&= \tfrac{1}{4}\Big|\big(\,\|x+y\| - \|-(x-y)\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| && \text{from (2)} \\
&\le \tfrac{1}{4}\Big|\big(\,\|(x+y) + (x-y)\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| && \text{from (18)} \\
&= \tfrac{1}{4}\Big|\big(\,\|2x\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| \\
&\le \tfrac{1}{4}\Big|\big(\,\|2x\|\,\big)\big(\,\|x\| + \|y\| + \|x\| + \|y\|\,\big)\Big| && \text{triangle inequality} \\
&= \tfrac{1}{4}\Big|\big(2\,\|x\|\,\big)\big(2\,\|x\| + 2\,\|y\|\,\big)\Big|
\end{aligned}
$$

Cancelling constants, we have

$$|\langle x, y\rangle| \;\le\; \|x\|\,\big(\,\|x\| + \|y\|\,\big)\,. \tag{19}$$

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

Now suppose $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, and $a_n \xrightarrow{n\to\infty} a \in \mathbb{R}$. We need to show that $f(a_n) \xrightarrow{n\to\infty} f(a)$ to establish that f is continuous on $\mathbb{R}$. Therefore, consider

$$
\begin{aligned}
|f(a_n) - f(a)| &= \Big|\langle a_n u, v\rangle - a_n\langle u, v\rangle - \Big(\langle au, v\rangle - a\langle u, v\rangle\Big)\Big| \\
&= \Big|\langle a_n u, v\rangle - \langle au, v\rangle - \Big(a_n\langle u, v\rangle - a\langle u, v\rangle\Big)\Big| \\
&= |\langle a_n u, v\rangle - \langle au, v\rangle - (a_n - a)\langle u, v\rangle| \\
&= |\langle a_n u, v\rangle + \langle -au, v\rangle - (a_n - a)\langle u, v\rangle| && \text{from (16)} \\
&= |\langle a_n u - au, v\rangle - (a_n - a)\langle u, v\rangle| && \text{from (12)} \\
&= |\langle (a_n - a)u, v\rangle - (a_n - a)\langle u, v\rangle| \\
&\le |\langle (a_n - a)u, v\rangle| + |(a_n - a)\langle u, v\rangle| \\
&= |\langle (a_n - a)u, v\rangle| + |a_n - a|\,|\langle u, v\rangle| \\
&\le \|(a_n - a)u\|\,(\|(a_n - a)u\| + \|v\|) + |a_n - a|\,|\langle u, v\rangle| && \text{from (19)} \\
&= |a_n - a|\Big(\|u\|\,\big(|a_n - a|\,\|u\| + \|v\| + |\langle u, v\rangle|\big)\Big) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

. . . continued . . .

Norms induced by inner products

Proof that $\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\|$ satisfies (PL).

Thus, $f(a_n) \xrightarrow{n\to\infty} f(a)$.

Since $a \in \mathbb{R}$ was arbitrary, f is continuous on $\mathbb{R}$.

Therefore, $f(a) = 0$ for all $a \in \mathbb{R}$, and hence, by definition (17), $\langle au, v\rangle = a\,\langle u, v\rangle$ for all $a \in \mathbb{R}$, as required.

Thus, it is indeed true that

$$\|x\| = \sqrt{\langle x, x\rangle} \iff \|\cdot\| \text{ satisfies (PL)}$$

i.e.,

A norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$ ***if and only if*** *$\|\cdot\|$ satisfies the parallelogram law (PL).*

$\square$

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The Final Exam

Final exam: What you need to know

- Everything discussed in class, including all definitions/concepts and theorems/lemmas/corollaries. It is essential that you understand how to use the definitions and theorems to construct proofs.
- Everything in assignments and the test. Solutions are posted on the course web site. *Make sure you fully understand all the solutions to all the problems in all the assignments and tests.*
- The TA's tutorial notes are also posted on the course web site.
- Most—but not all—of the material that you are responsible for is covered in chapters 7–9 and 11 in the BS textbook, and/or chapters 7–10 and 13 in the TBB textbook. You are not responsible for material in the textbook that was not covered in lectures or assignments, but there are many problems in these chapters that we did not discuss and would provide excellent practice for the exam.

Final exam: other information

- My office hours will be different during the exam period.
  - Thursday 10 April 2025 @ 1:30–2:30 pm
  - Thursday 17 April 2025 @ 1:30–2:30 pm
  - Wednesday 23 April 2025 @ 10:00 am to 12:00 pm and 2:00–4:00 pm (possibly online; to be confirmed: check course web site)
- The above dates and times are tentative. Check the course web site for announcements about any changes in office hours during the exam period.
- Check Final exam page on course web site for any further info.
- Some additional problems will be posted in the coming days. *It is possible that one or more of those problems—or a very similar problem—might appear on the final exam.*
- I will post a blank exam at the end of this week or early next week, so you can see the structure of the exam (similar in style to the test structure document I shared before the midterm).

Other courses you might enjoy if you liked Math 3A03

- **Math 4A03** Real Analysis II
  - Lebesgue theory of measure and integration, and other topics
- **Math 4AT3*** Topics in Analysis
- **Math 4L03*** Introduction to Mathematical Logic
- **Math 3F03** Ordinary Differential Equations
  - Qualitative theory of ODEs / dynamical systems
  - equilibria, stability, attractors
  - *prerequisite:* Math 2C03
- **Math 4MB3** Mathematical Biology
  - analysis and ODE theory applied to epidemic modelling
  - *prerequisite:* Math 3F03 (or permission of instructor)
- **Math 3DC3*** Discrete Dynamical Systems and Chaos
- **Stats 3PG3** Probability and Games of Chance

Thank you!

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