33 What is $\mathbb{R}$?

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McMaster University

Mathematics and Statistics

$$\int_M d\omega = \int_{\partial M} \omega$$

Mathematics 3A03
Real Analysis I

Instructor: David Earn

Lecture 33

What is $\mathbb{R}$?

Friday 4 April 2025

Announcements

- Participation deadline for Assignment 5 was 1:25pm today.
- Solutions to Assignment 5 are now posted.
- In the slides from Wednesday's lecture, the end of the proof that "continuity on a compact set implies uniform continuity" has been corrected and improved. Make sure to download the updated version.
- New, exciting topic today...

What exactly is $\mathbb{R}$?

Poll

- Go to
  https://www.childsmath.ca/childsa/forms/main_login.php
- Click on Math 3A03
- Click on Take Class Poll
- Fill in poll **Construction of Reals**
- `Submit`.

Informal introduction to construction of numbers ($\mathbb{N}$)

- Assume we know what a ***set*** is.
- Define $0 \equiv \varnothing = \{\}$ (the empty set)
- Define $1 \equiv \{0\} = \{\varnothing\} = \{\{\}\}$
- Define $2 \equiv \{0, 1\} = \{\{\}, \{\{\}\}\}$
- Define $n + 1 \equiv n \cup \{n\}$ (successor function)
- Define ***natural numbers*** $\mathbb{N} = \{1, 2, 3, \dots\}$
- Thus, n is defined to be a set containing n elements.

Informal introduction to construction of numbers ($\mathbb{N}$)

Historical note:

- We have defined n to be a set containing n elements.
- Logicians first tried to define n as "the set of all sets containing n elements".
- The earlier definition possibly better captures our intuitive notion of what n "really is", but such "sets" are unweildy and create serious challenges for development of mathematical foundations.

Informal introduction to construction of numbers ($\mathbb{N}$)

Order of natural numbers:

- Natural numbers defined as above have the right order:

$$m \leq n \quad \iff \quad m \subseteq n$$

Note: we *define* "$\leq$" on natural numbers via "$\subseteq$" on sets.

Addition and multiplication of natural numbers:

- Still possible to define in terms of sets, but trickier.
  See this free e-book:
  "Transition to Higher Mathematics"
  http://openscholarship.wustl.edu/books/10/.

Informal introduction to construction of numbers ($\mathbb{Z}$)

Integers:

- Need additive inverses for all natural numbers.
- Need to define $\cdot$, $+$, $-$, for all pairs of integers.
- Again, possible to define everything via set theory.

- We'll assume we "know" what the naturals $\mathbb{N}$ and the integers $\mathbb{Z}$ "are".
- We can then *construct* the rationals $\mathbb{Q}$...

Informal introduction to construction of numbers ($\mathbb{Q}$)

Rationals:

- *Idea:* Associate $\mathbb{Q}$ with $\mathbb{Z} \times \mathbb{N}$
- Use notation $\frac{a}{b} \in \mathbb{Q}$ if $(a, b) \in \mathbb{Z} \times \mathbb{N}$.
- Define equivalence of rational numbers:

$$\frac{a}{b} = \frac{c}{d} \quad \stackrel{\textbf{def}}{=} \quad a \cdot d = b \cdot c$$

- Define order for rational numbers:

$$\frac{a}{b} \leq \frac{c}{d} \quad \stackrel{\textbf{def}}{=} \quad a \cdot d \leq b \cdot c$$

Informal introduction to construction of numbers ($\mathbb{Q}$)

Rationals, continued:

- Define operations on rational numbers:

$$\frac{a}{b} + \frac{c}{d} \stackrel{\text{def}}{=} \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} \stackrel{\text{def}}{=} \frac{a \cdot c}{b \cdot d}$$

- Constructed in this way (ultimately from the empty set), $\mathbb{Q}$ satisfies all the standard properties that we associate with the rational numbers.

- Formally, $\mathbb{Q}$ is a set of equivalence classes of $\mathbb{Z} \times \mathbb{N}$.
Extra Challenge Problem: Are "$+$" and "$\cdot$" well-defined on $\mathbb{Q}$?

Construction of the Real Numbers

- Recall that we defined the natural numbers $\mathbb{N}$ as sets: $0 \equiv \varnothing,\ \ 1 \equiv \{0\},\ \ 2 \equiv \{0,1\},\ \ etc.$
- For $m,n \in \mathbb{N}$ we defined $m < n$ to mean $m \subset n$.
- We defined the rational numbers $\mathbb{Q}$ to be ordered pairs of integers (more precisely, $\mathbb{Q}$ is a set of equivalence classes of $\mathbb{Z} \times \mathbb{N}$).
- In the same spirit, we can define real numbers not by determining what they "really are" but instead by settling for a definition that determines their mathematical properties completely.
- So, just as $\mathbb{Z}$ can be built from $\mathbb{N}$, and $\mathbb{Q}$ can be built from $\mathbb{Z}$, we can build $\mathbb{R}$ from $\mathbb{Q}$.
- Richard Dedekind's idea was to construct a real number α as a set of rational numbers, in a way that naturally yields the one property of $\mathbb{R}$ that $\mathbb{Q}$ does not have: least upper bounds...

Construction of the Real Numbers

Dedekind's stroke of genius (on 24 Nov 1858) was to define α as "*the set of rational numbers less than α*" in a way that is not circular.

Definition (Real number)

A ***real number*** is a set $\alpha \subseteq \mathbb{Q}$, with the following four properties:

1. $\forall x \in \alpha$, if $y \in \mathbb{Q}$ and $y < x$, then $y \in \alpha$
   i.e., α is ***downward closed***;
2. $\alpha \neq \varnothing$;
3. $\alpha \neq \mathbb{Q}$;
4. there is ***no greatest element*** in α,
   i.e., if $x \in \alpha$ then $\exists\, y \in \alpha$ such that $y > x$.

The set of all real numbers is denoted by $\mathbb{R}$.

Historical note: Dedekind originally defined a real number α as the pair of sets (L, R), where L is the set of rationals $< \alpha$ and R is the set of rationals $\geq \alpha$. A real number is then described as a ***Dedekind cut***.

Construction of the Real Numbers

Example: $\sqrt{2} = \{q \in \mathbb{Q} : q^2 < 2 \text{ or } q < 0\}$.

With real numbers defined, we can easily define an ordering on $\mathbb{R}$.

Definition (Order of real numbers)

If $\alpha, \beta \in \mathbb{R}$ then $\alpha < \beta$ iff $\alpha \subset \beta$.
(Similarly for $>$, $\leq$, and $\geq$.)

We now have enough to prove:

Theorem ($\mathbb{R}$ is complete)

If $A \subset \mathbb{R}$, $A \neq \varnothing$, and A is bounded above, then A has a least upper bound.

We also need to define $+$, $\cdot$, 1 and α^{-1}.

Then we can prove that $\mathbb{R}$ *is a complete ordered field* and, moreover, it is the *unique* such field (up to isomorphism).

Construction of the Real Numbers

Proof that $\mathbb{R}$ is complete.

Suppose $A \subset \mathbb{R}$, $A \neq \varnothing$, and A is bounded above.

Let $\beta = \{x : x \in \alpha \text{ for some } \alpha \in A\} = \bigcup_{\alpha \in A} \alpha$.

Since each $x \in \beta$ is in some set $\alpha \subseteq \mathbb{Q}$, we have $\beta \subseteq \mathbb{Q}$.
To verify that $\beta \in \mathbb{R}$, we check the four defining properties:

1. Suppose (i) $x \in \beta$ and (ii) $y < x$. (i) $\implies$ $x \in \alpha$ for some $\alpha \in A$. But α is a real number, so (ii) $\implies$ $y \in \alpha$. Hence $y \in \beta$.
2. Since $A \neq \varnothing$, $\exists \alpha \in A$. Since α is a real number, $\exists x \in \alpha$. This implies $x \in \beta$, so $\beta \neq \varnothing$.
3. Since A is bounded above, there is some real number γ such that $\alpha < \gamma$ for every $\alpha \in A$. Since γ is a real number, there is some rational number $x \notin \gamma$. But $\alpha < \gamma$ means that $\alpha \subset \gamma$, so it follows that $x \notin \alpha$ for any $\alpha \in A$. This implies $x \notin \beta$, so $\beta \neq \mathbb{Q}$.

...continued...

Construction of the Real Numbers

Proof that $\mathbb{R}$ is complete (*continued*).

4. Suppose $x \in \beta$. Then $x \in \alpha$ for some $\alpha \in A$. Since α does not have a greatest element, $\exists y \in \mathbb{Q}$ with $x < y$ and $y \in \alpha$. But this implies $y \in \beta$; thus β does not have a greatest element.

These four points establish that β is a real number. It remains to show that β is the least upper bound of A.

If $\alpha \in A$, then $\alpha \subseteq \beta$, *i.e.*, $\alpha \leq \beta$, so β is an upper bound for A. On the other hand, if γ is an upper bound for A, then $\alpha \leq \gamma$ for every $\alpha \in A$; this implies $\alpha \subseteq \gamma$, for every $\alpha \in A$, and hence $\beta \subseteq \gamma$, *i.e.*, $\beta \leq \gamma$. Thus β is the <u>least</u> upper bound of A. □

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